Note: You must indicate the information source which you use.

Todays topics:

**Proposition.** Let $a \in \mathbb{R}_{>0}$, $b \in \mathbb{R}$. Let $X$ be a continuous random variable with a density function $f_X$, and let $Y := aX + b$, then the density function $f_Y$ of $Y$ is
\[
  f_Y(t) = \frac{1}{a} f_X \left( \frac{t - b}{a} \right).
\]

**Observation.** Let $X, Y$ be continuous random variables which are independent, and let $f_X, f_Y$ be their probability functions, respectively. Let $Z := X + Y$, then the probability function $f_Z$ of $Z$ is
\[
  f_Z(t) = \int_{-\infty}^{+\infty} f_X(s) f_Y(t - s) ds
\]

**Definition.**
(i) Let $X$ be a nonnegative integer valued random variable, with a probability function $f$, then the probability generating function is defined by
\[
  g(z) \overset{\text{def.}}{=} E[z^X] = \sum_{k=0}^{+\infty} z^k f(k) \quad (-1 < z < 1).
\]
(ii) Let $X$ be a random variable, then the moment generating function is defined by
\[
  M(\theta) \overset{\text{def.}}{=} E[e^{\theta X}] \quad (\theta \in \mathbb{R}).
\]
(iii) Let $X$ be a random variable, then the characteristic function is defined by
\[
  \varphi(t) \overset{\text{def.}}{=} E[e^{itX}] \quad (t \in \mathbb{R})
\]
where $i$ denotes the imaginary unit.

**Proposition.**
(i) $g(1) = 1$. $g'(1) = E[X]$. $g''(1) = E[X^2] - E[X]$.
(ii) $M^{(k)}(0) = E[X^k]$.
(iii) $\varphi^{(k)}(0) = E[(iX)^k]$.

**Theorem.** Let $X, Y$ be independent random variables, and let $M_X(\theta)$, $M_Y(\theta)$ be their moment generating functions, respectively. Let $Z := X + Y$, then its moment generating function $M_Z(\theta)$ satisfies $M_Z(\theta) = M_X(\theta)M_Y(\theta)$.

**Theorem.** Let $X, Y$ be independent random variables according to distribution functions $F_X$ and $F_Y$, respectively, and let $M_X(\theta)$, $M_Y(\theta)$ be their moment generating functions, respectively. If $M_X(\theta) = M_Y(\theta)$, then $F_X = F_Y$. 
Exercises (try at least two exercises. * is fundamental.)

Ex. 1. Compute the moment generating function of the following distributions.
   (i) Bernoulli distribution B(1; p).
   (ii) Binomial distribution B(n; p).
   (iii) Geometric distribution Ge(p).
   (*iv) Poisson distribution Po(λ).
   (v) Exponential distribution Ex(α).
   (*vi) Normal distribution N(μ, σ²).

*Ex. 2. Let X, Y are independent random variables such that X ~ G(α, ν₁), Y ~ G(α, ν₂), where ν₁ and ν₂ are positive integers. Show the density function of Z := X + Y.

*Ex. 3. Let X, Y are independent random variables where X ~ N(μ₁, σ₁²), Y ~ N(μ₂, σ₂²). Show the density function of Z := X + Y.

Ex. 4.
   (i) Let a ∈ R⁺, b ∈ R, and let X ~ N(μ, σ²). Show the density function of Y := aX + b.
   (ii) Let X₁, . . . , Xₙ are random variables i.i.d., with a (common) expectation μ and a (common) variance σ². Show the density function of Z := (X₁ + · · · + Xₙ)/n.
   Hint. use central limit theorem.