Graduate School of ISEE

Note: You must indicate the information source which you use.

Today’s topics:

**Definition** $T$ is an *consistent estimator* of $\theta$ if $\lim_{n \to +\infty} \Pr([T] = \theta) = 1$.

**Definition** $T$ is an *unbiased estimator* of $\theta$ if $\operatorname{E}[T] = \theta$.

**Definition** For an estimator $T$ of $\theta$, $\operatorname{E}[(T - \theta)^2]$ is called *mean square error*.

Review:

**Proposition** Suppose $X, Y$ are independent continuous random variables, where $f_X$ and $f_Y$ are probability density function of $X$ and $Y$. Then, the joint density function $f_{XY}(x, y)$ of $X$ and $Y$ satisfies $f_{XY}(x, y) = f_X(x)f_Y(y)$.

Exercises

*Ex 1.* Let $X_1, \ldots, X_n$ be i.i.d., then show that $\frac{1}{n-1} \sum_{i=1}^{n} (X_i - \bar{X})^2$ is an unbiased estimator of population variance, where $\bar{X} := \frac{1}{n} \sum_{i=1}^{n} X_i$.

*Ex 2.* Prove the Proposition.

†Ex. 3. Let $X_1, \ldots, X_n$ are independent random variables each of which is according to $\text{N}(0, 1)$. Let $Z := X_1^2 + \cdots + X_n^2$, then show that $Z \sim \text{G}(1/2, n/2)$. Remember that the density function of the Gamma distribution $\text{G}(\alpha, \nu)$ ($\alpha > 0, \nu > 0$) is defined by

$$f(x) = \begin{cases} \frac{1}{\Gamma(\nu)} \alpha^\nu x^{\nu-1} e^{-\alpha x} & (x \geq 0), \\ 0 & (x < 0), \end{cases}$$

where $\Gamma(\nu) = \int_0^{+\infty} t^{\nu-1} e^{-t} dt$.

†Ex. 4. Let $X, Y$ are independent random variables where $X \sim \text{N}(0, 1)$ and $Y \sim \text{G}(1/2, n/2)$. Let $T := X/\sqrt{Y/n}$, then show that the density function of $T$ is described as

$$f_T(t) = \frac{\Gamma((n+1)/2)}{\sqrt{n\pi}\Gamma(n/2)} \left(1 + \frac{t^2}{n}\right)^{-(n+1)/2} \quad (-\infty < t < +\infty)$$