Finding a Path in Group-Labeled Graphs with Two Labels Forbidden

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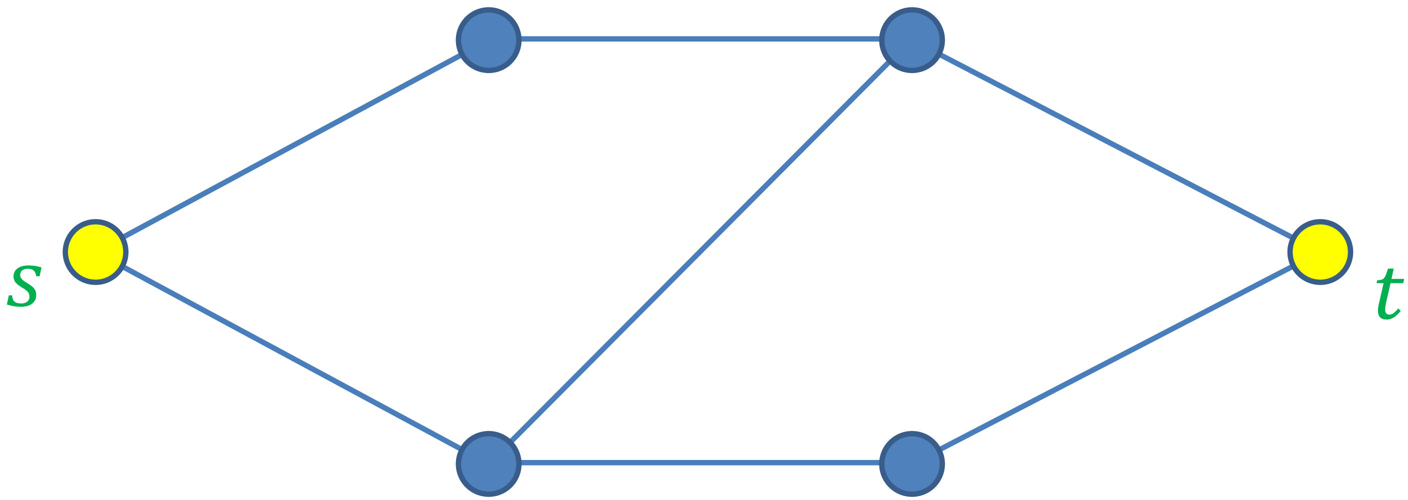
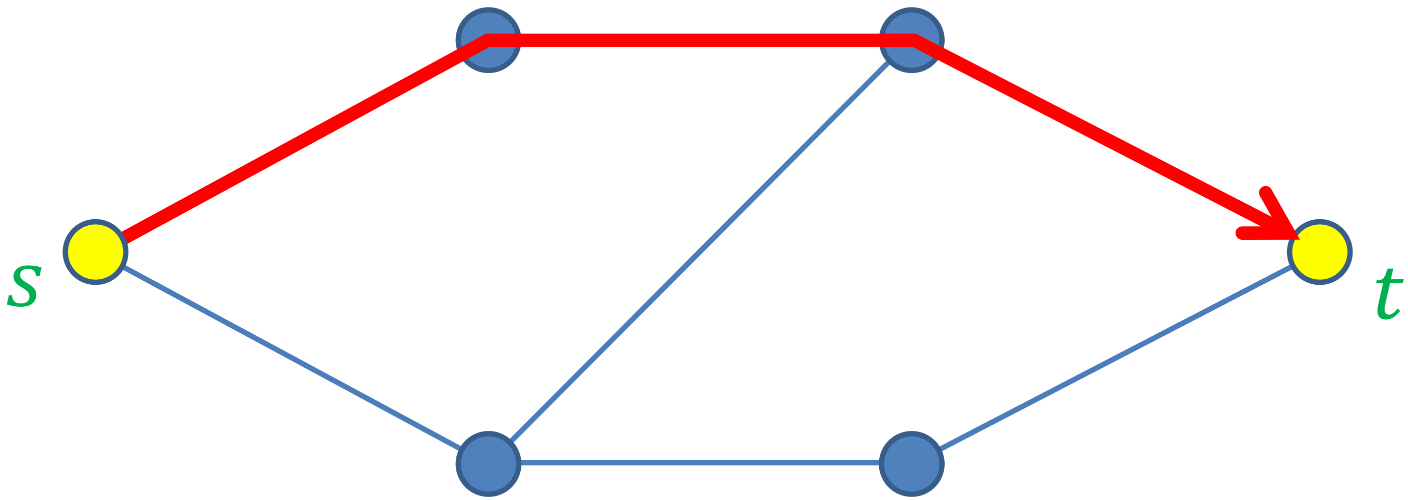
Parity of Length



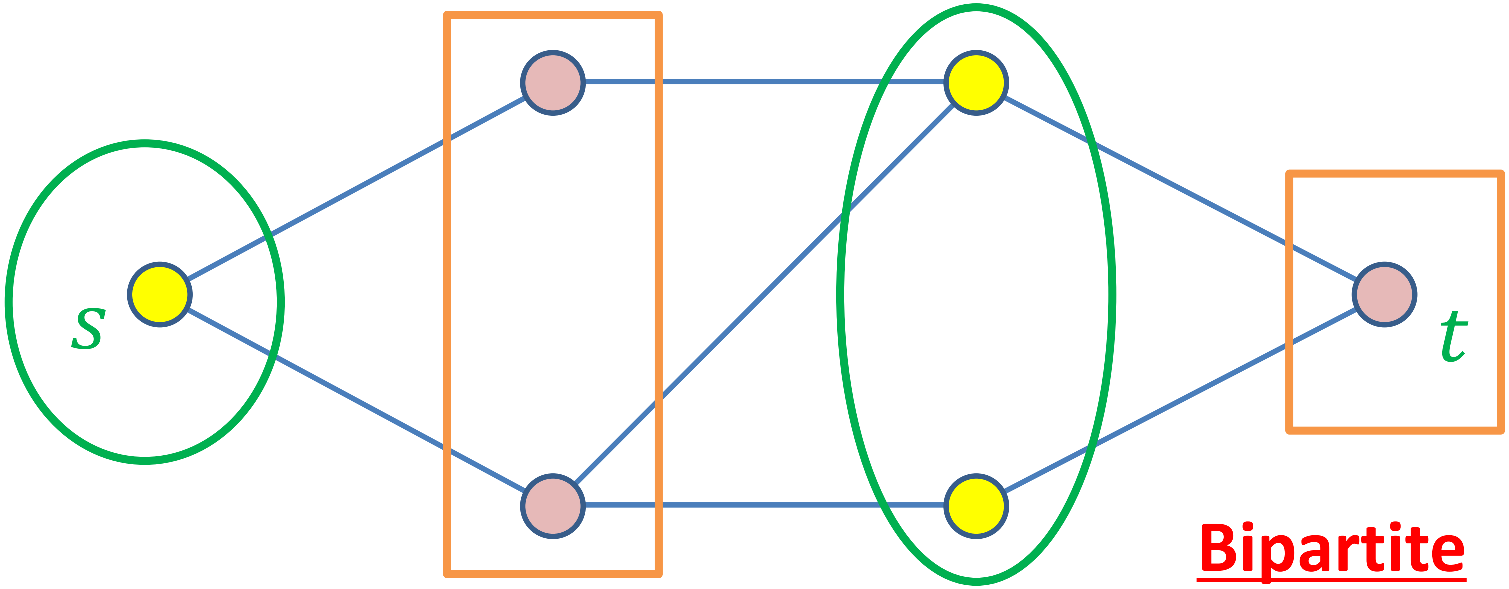
- Odd → ???
- Even → ???

Parity of Length



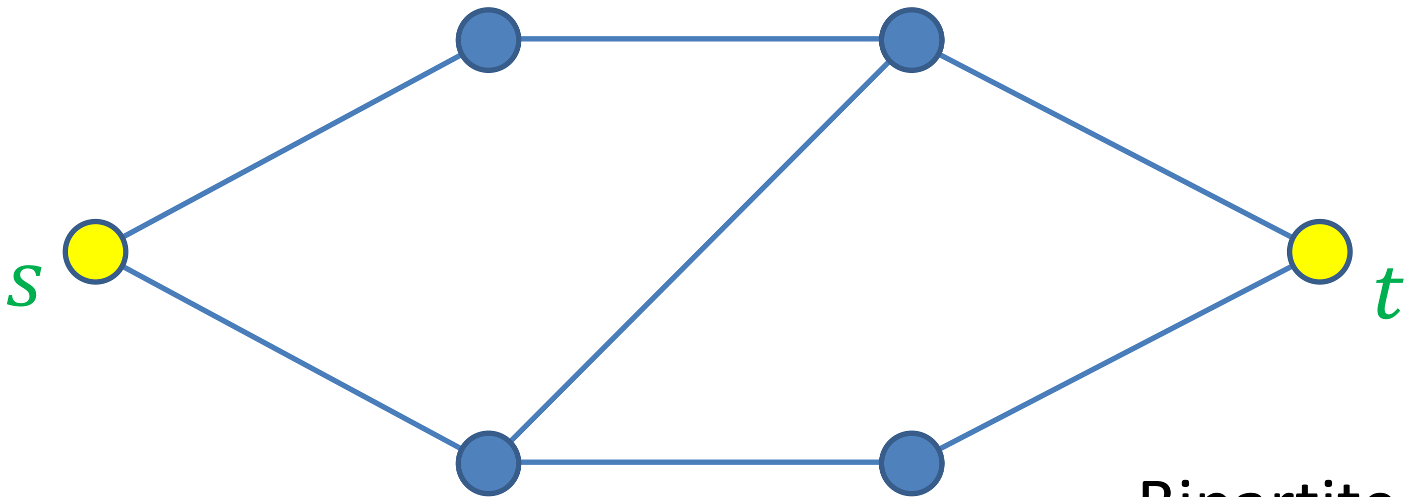
- **<u>Odd → YES</u>**
- Even → ???

Parity of Length



- Odd $\rightarrow$ YES
- **Even $\rightarrow$ NO**

Parity of Length



Bipartite

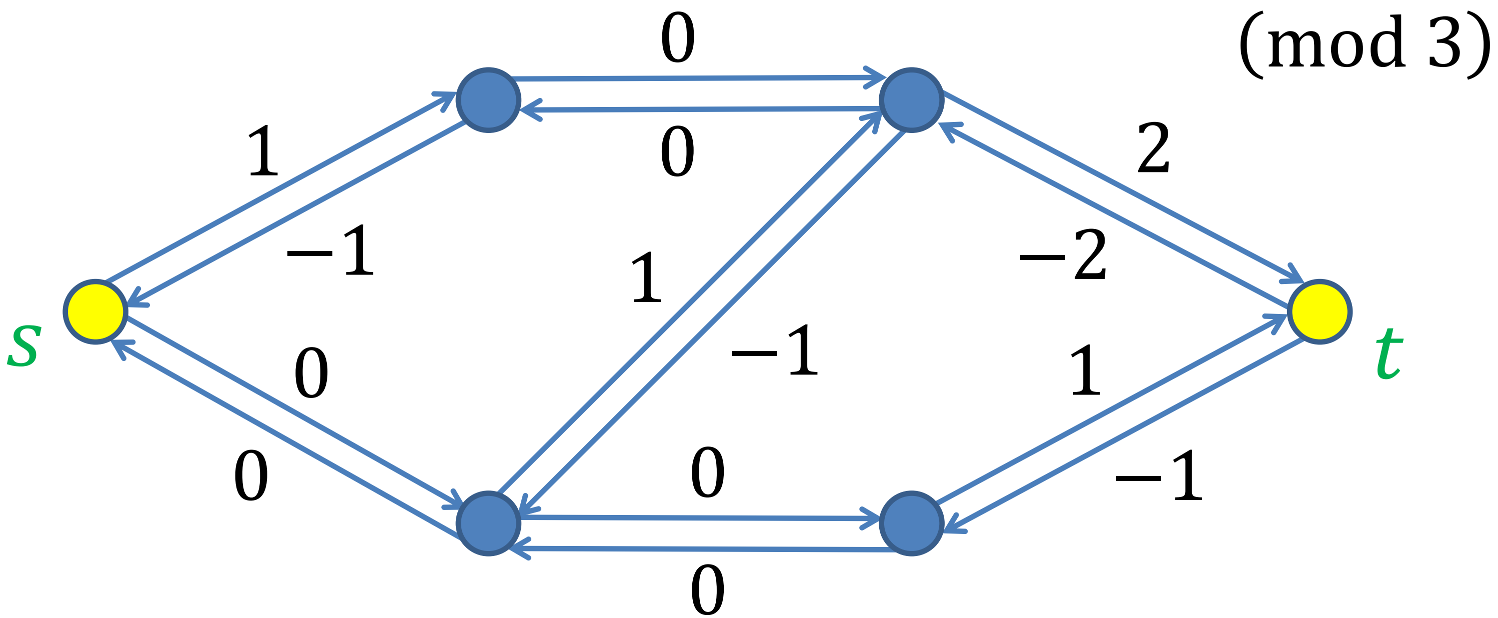
↓

Easy to test!

- Odd → YES
- Even → NO

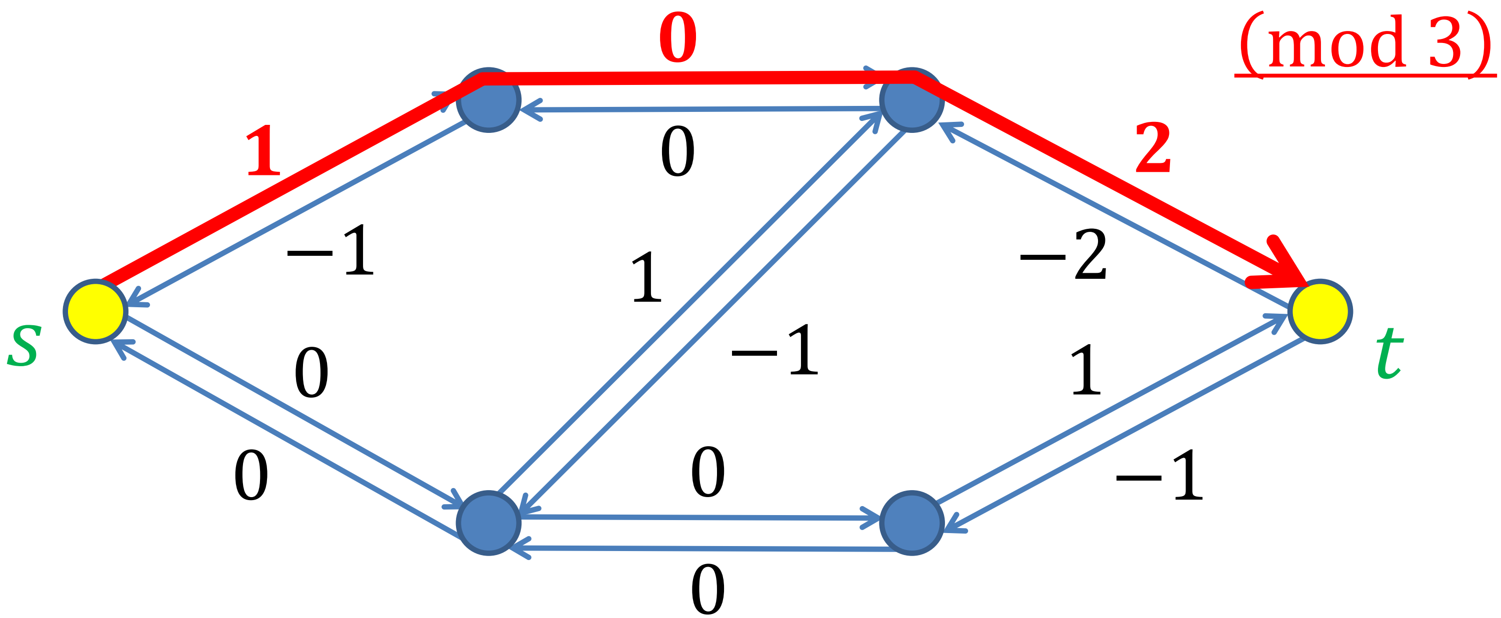
Parity of s–t paths $= \{$Odd$\}$

Possible Labels



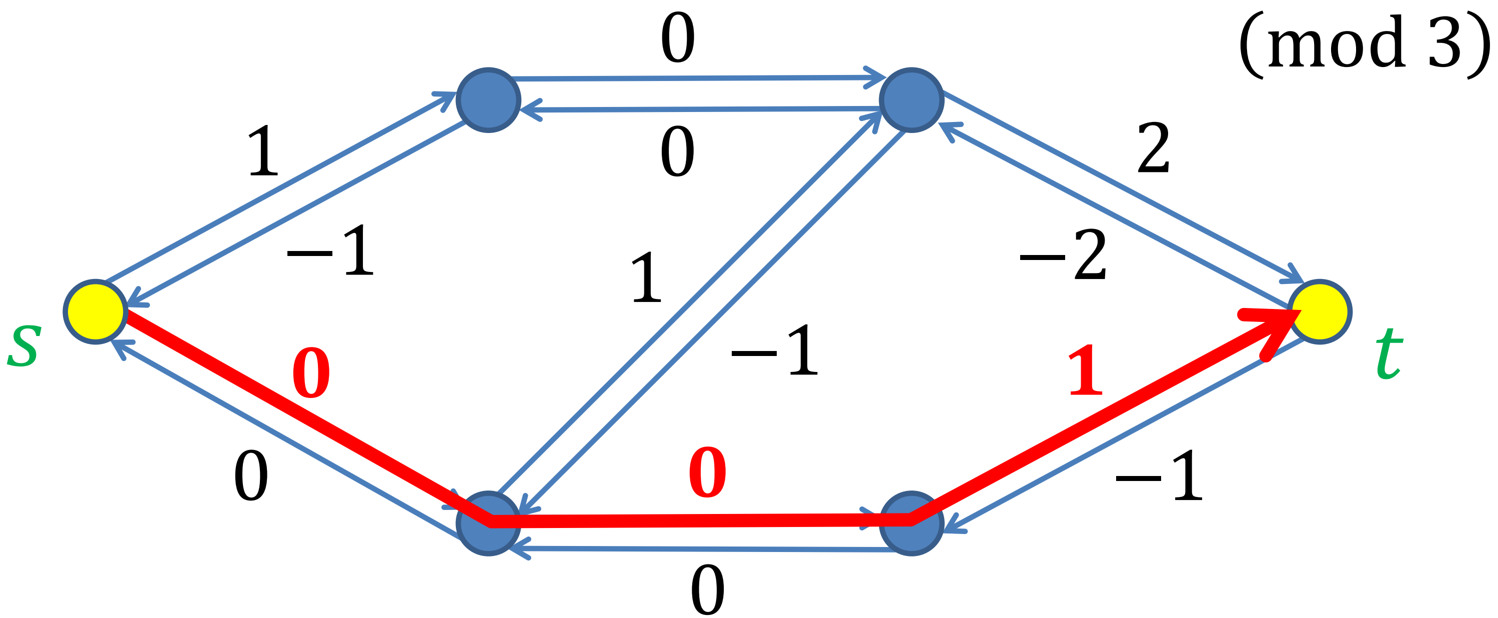
- $0 \rightarrow$???
- $1 \rightarrow$???
- $2 \rightarrow$???

Possible Labels



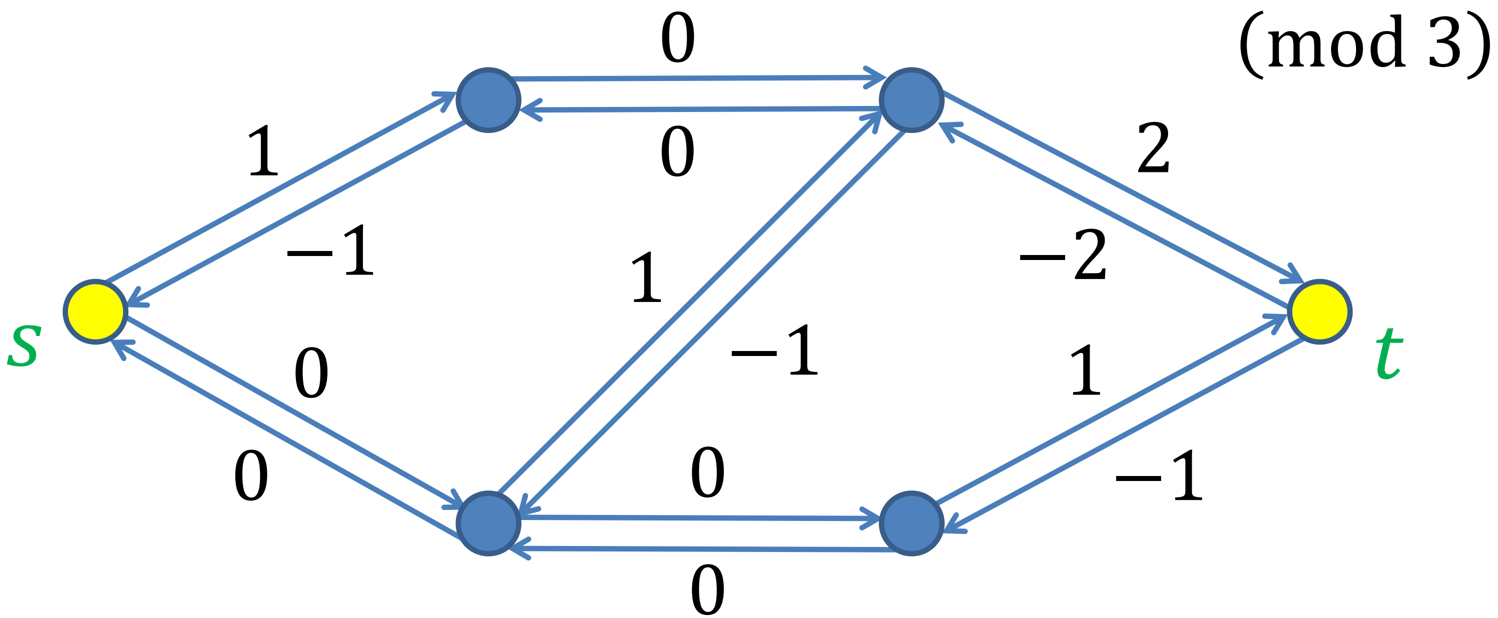
- $\underline{0 \rightarrow \textbf{YES}}$ $1 + 0 + 2 = 3 \equiv 0$
- $1 \rightarrow ???$
- $2 \rightarrow ???$

Possible Labels



- $0 \;\rightarrow\;$ YES
- <u>$1 \;\rightarrow\;$ **YES**</u> $\quad 0 + 0 + 1 = 1$
- $2 \;\rightarrow\;$???

Possible Labels



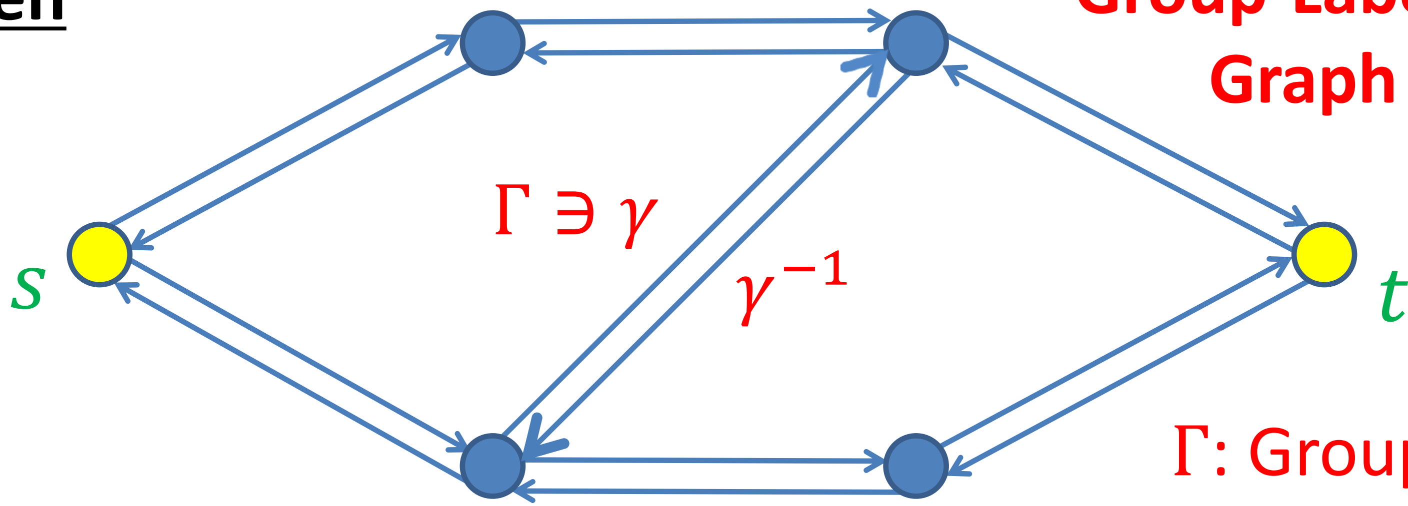
- $0 \rightarrow$ YES
- $1 \rightarrow$ YES
- $2 \rightarrow$ NO

Labels of s–t paths $= \{0, 1\}$

Our Problem

Given

Group-Labeled Graph

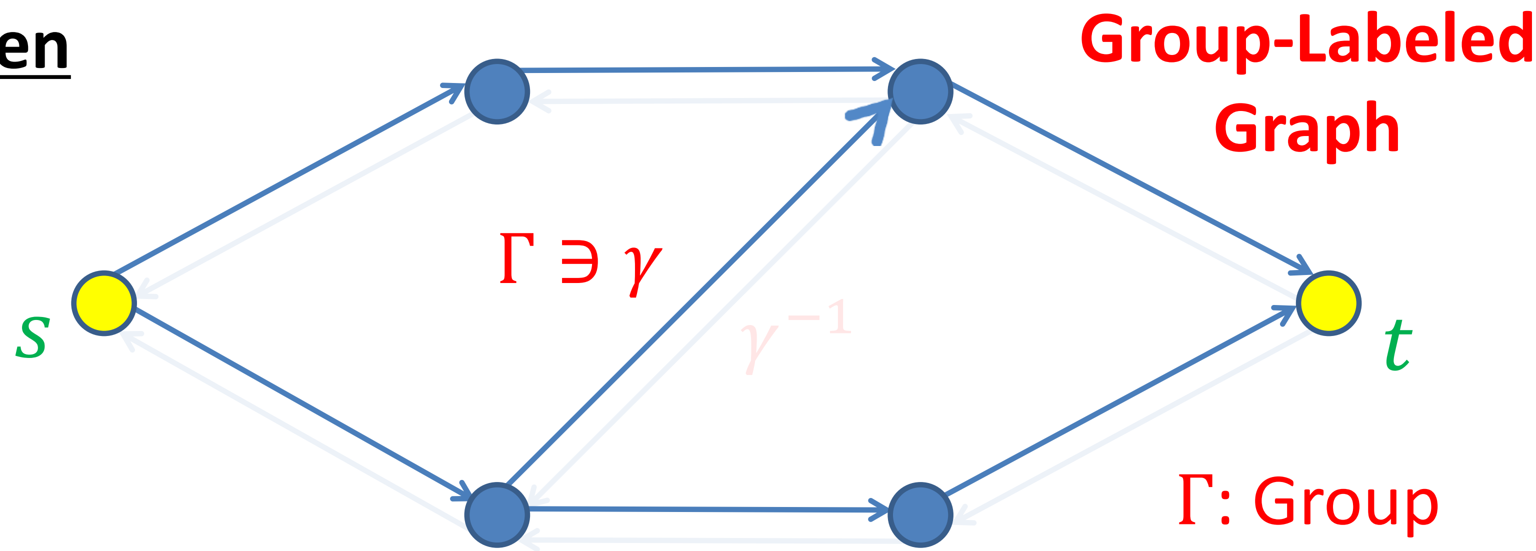


Γ: Group

Find **Possible Labels of s–t paths**

Our Problem

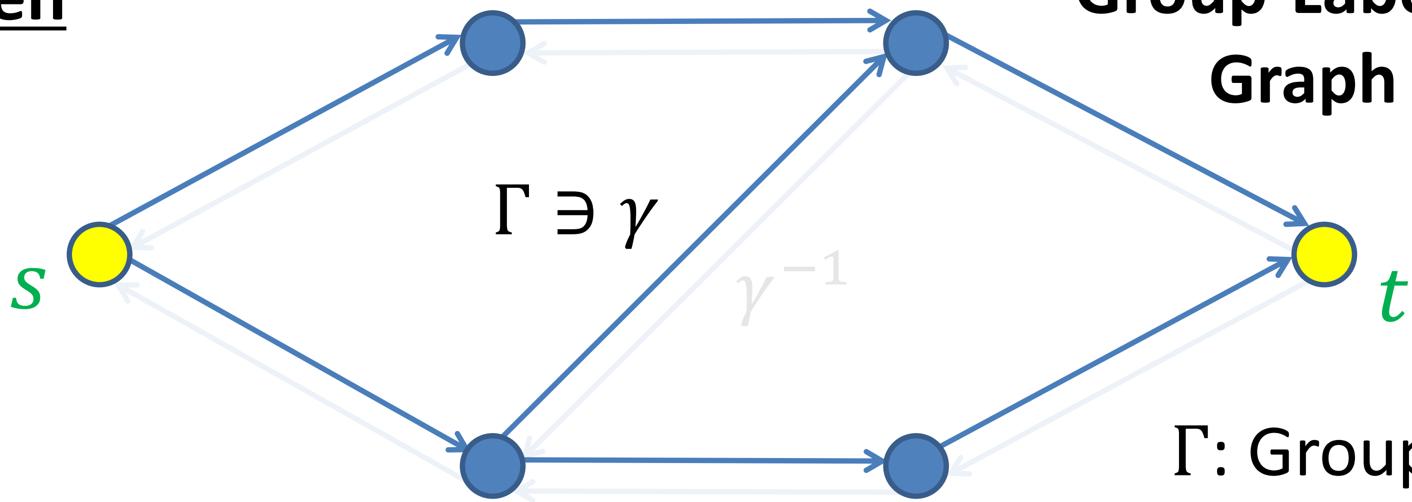
Given



Find **Possible Labels of s–t paths**

Our Problem

Given



Group-Labeled Graph

Γ: Group

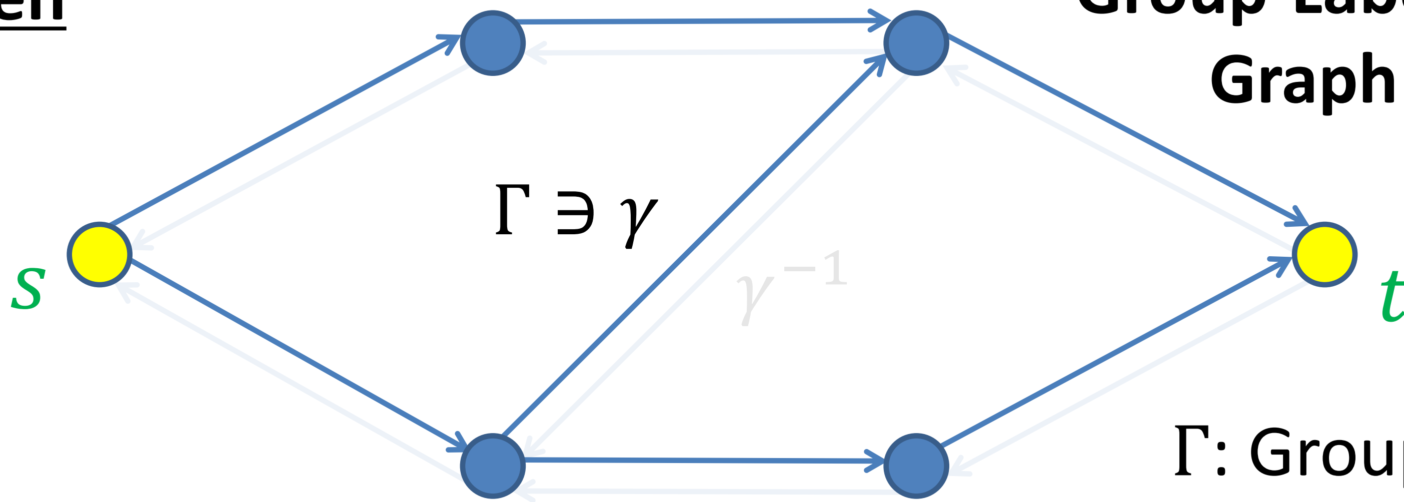
Find **Possible Labels of s–t paths** → **Difficult**

- $l = \{\alpha\}$: Polytime
- $l \supseteq \{\alpha\}$: NP-hard (Hamiltonian Path)
- $l = \{\alpha, \beta\}$: ???

Our Problem

<ins>**Given**</ins>

Group-Labeled Graph



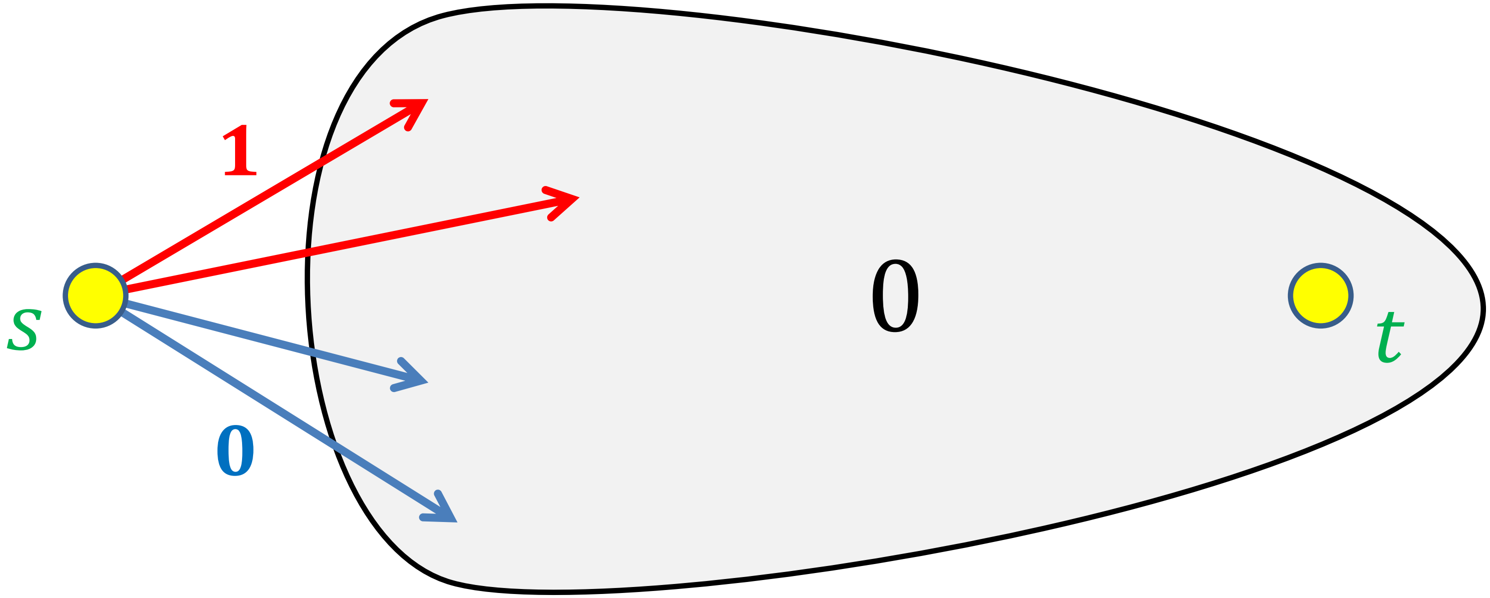
Γ: Group

<ins>**Find**</ins> **Possible Labels of s–t paths** → Difficult

- $l = \{\alpha\}$: Polytime
- $l \supseteq \{\alpha\}$: NP-hard (Hamiltonian Path)
- <ins>$l = \{\alpha, \beta\}$: **Polytime!!**</ins>

Example 1

Z-Labeled Graph
$\mathbf{Z} = \{0, \pm 1, \pm 2, \dots\}$

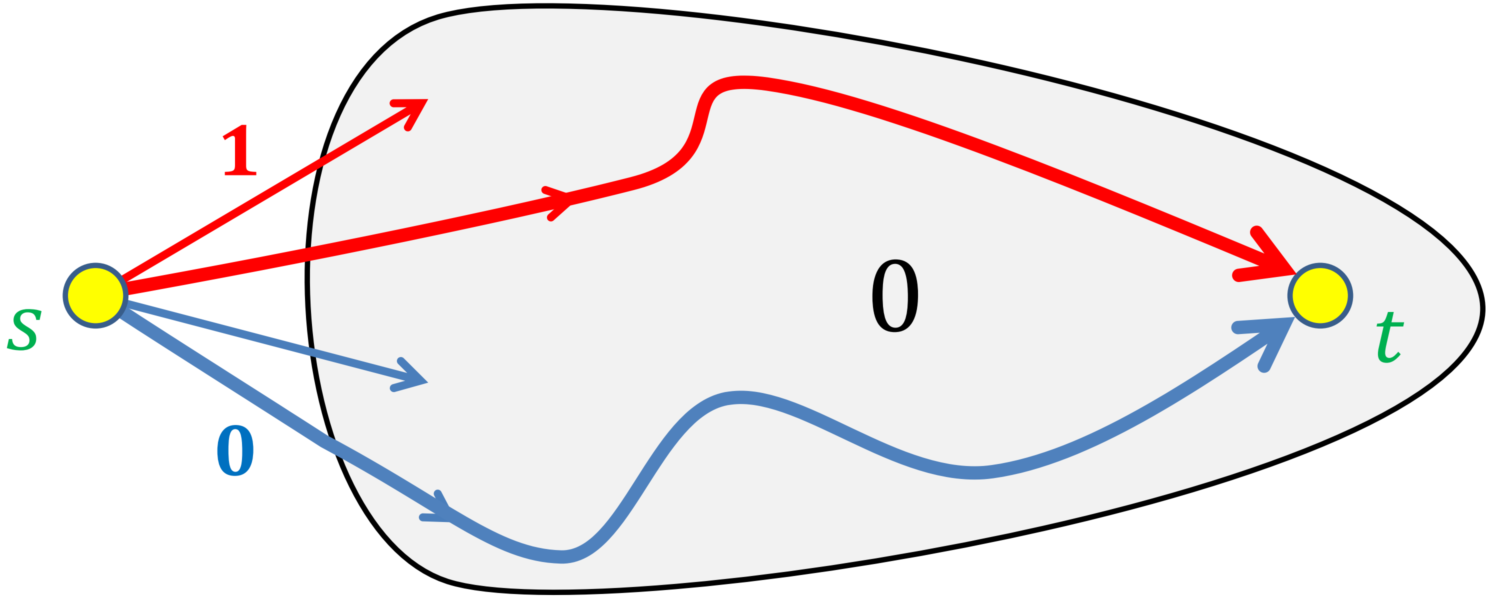


$l = \{0, 1\}$

Example 1

$\mathbf{Z}$-Labeled Graph

$\mathbf{Z} = \{0, \pm 1, \pm 2, \dots\}$

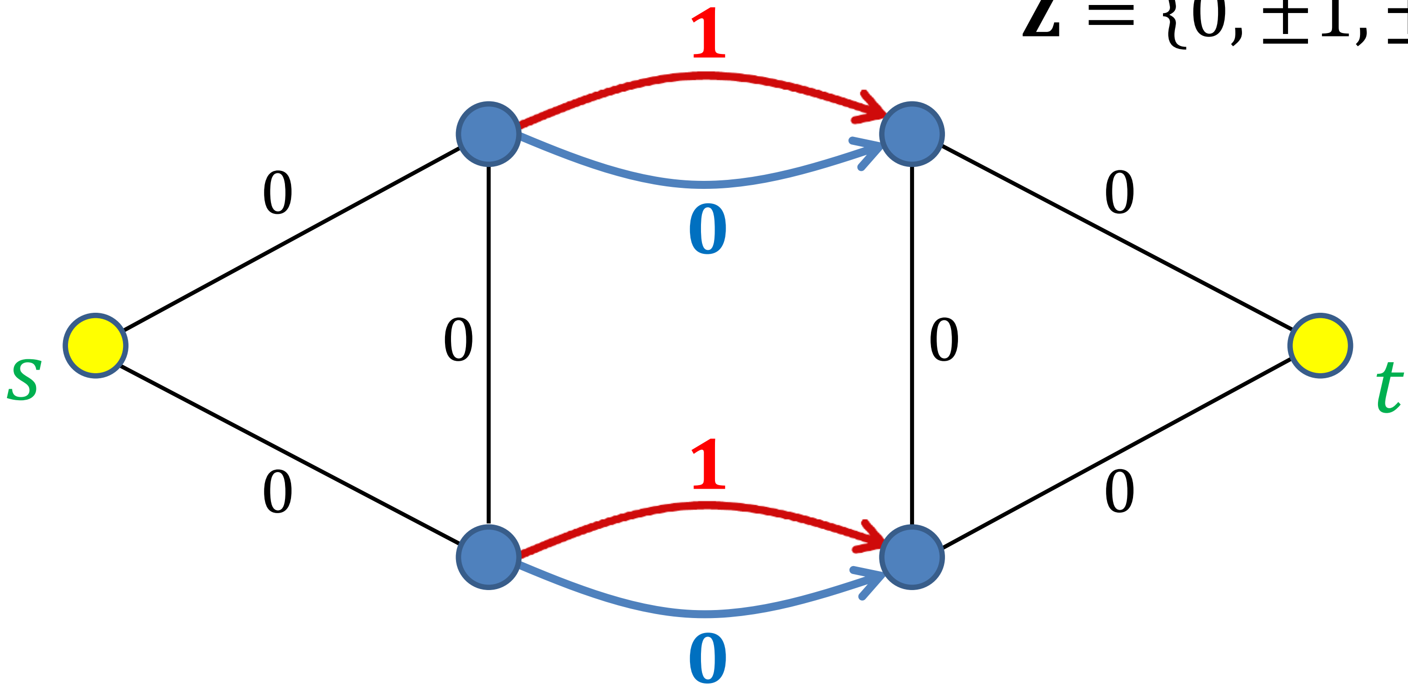


$l = \{0, 1\}$

Example 2

$\mathbf{Z}$-Labeled Graph

$\mathbf{Z} = \{0, \pm 1, \pm 2, \dots\}$

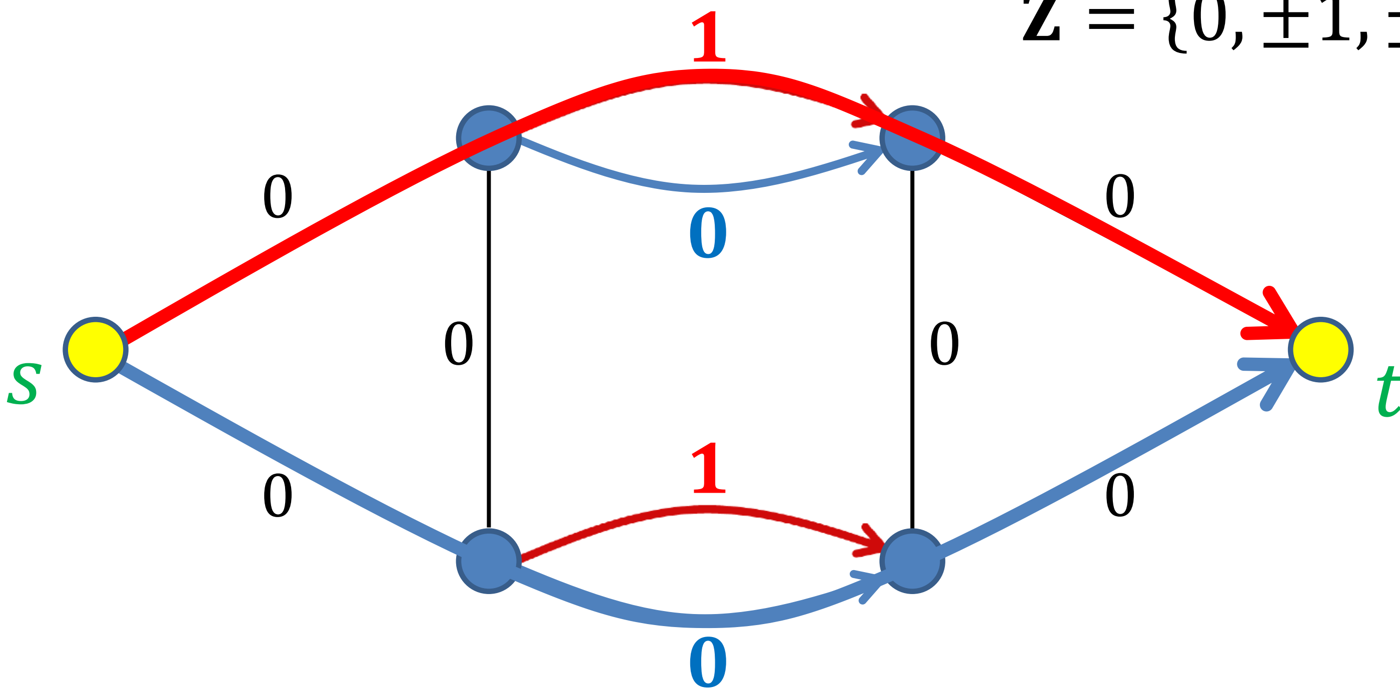


$l = \{0, 1\}$

Example 2

Z-Labeled Graph

$\mathbf{Z} = \{0, \pm 1, \pm 2, \ldots\}$

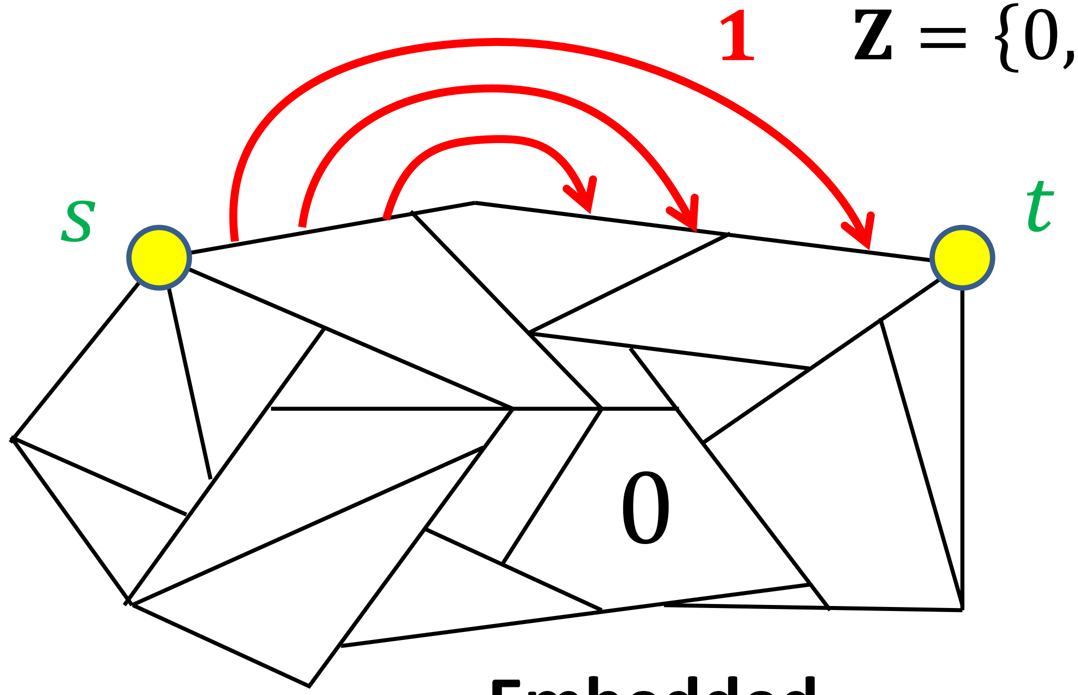


$l = \{0, 1\}$

Example 3

Z-Labeled Graph

$\mathbf{Z} = \{0, \pm 1, \pm 2, \ldots\}$



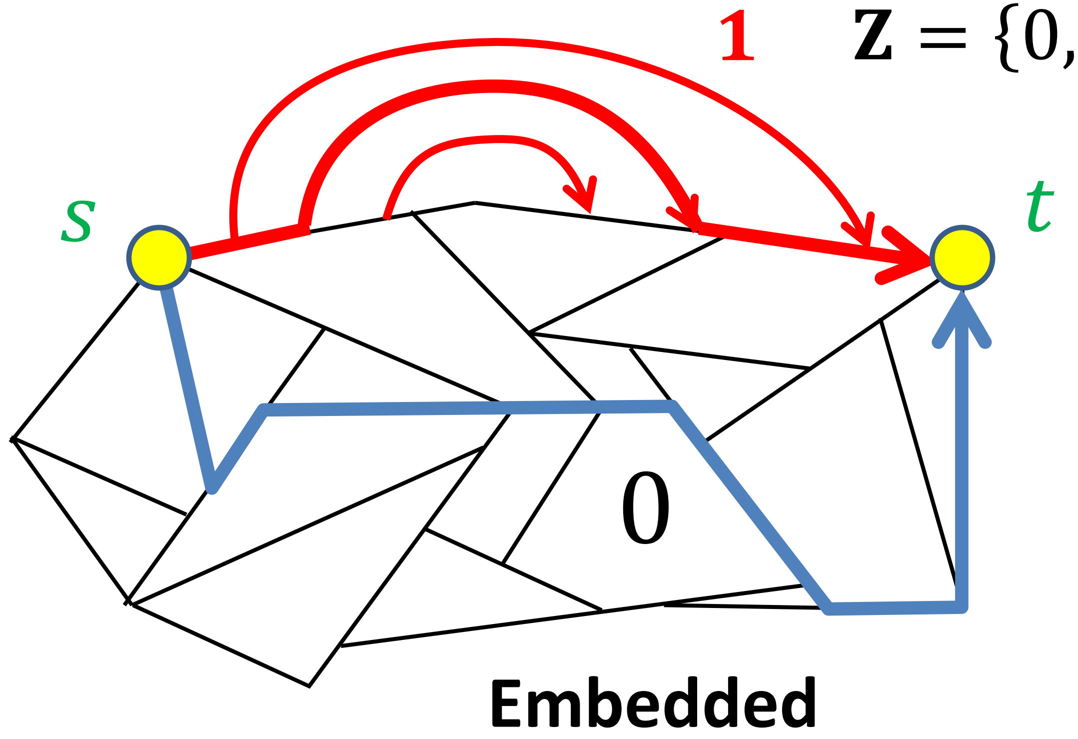
Embedded

$l = \{0, 1\}$

Example 3

Z-Labeled Graph

$\mathbf{Z} = \{0, \pm 1, \pm 2, \dots\}$



$l = \{0, 1\}$

Example 3

$\mathbf{Z}$-Labeled Graph

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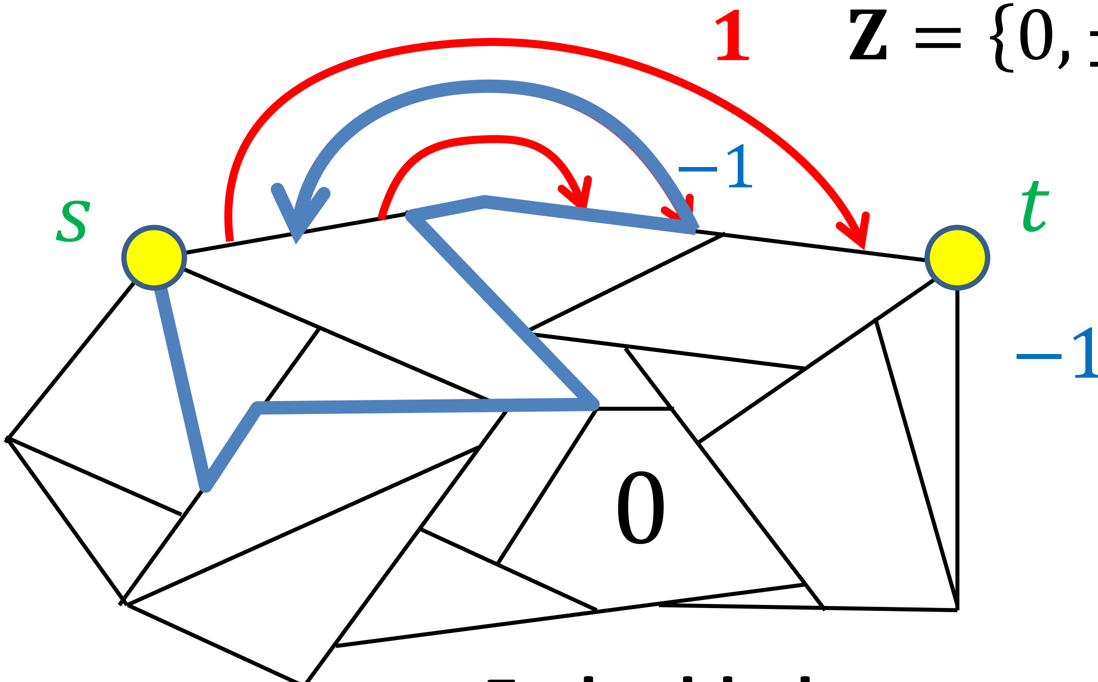
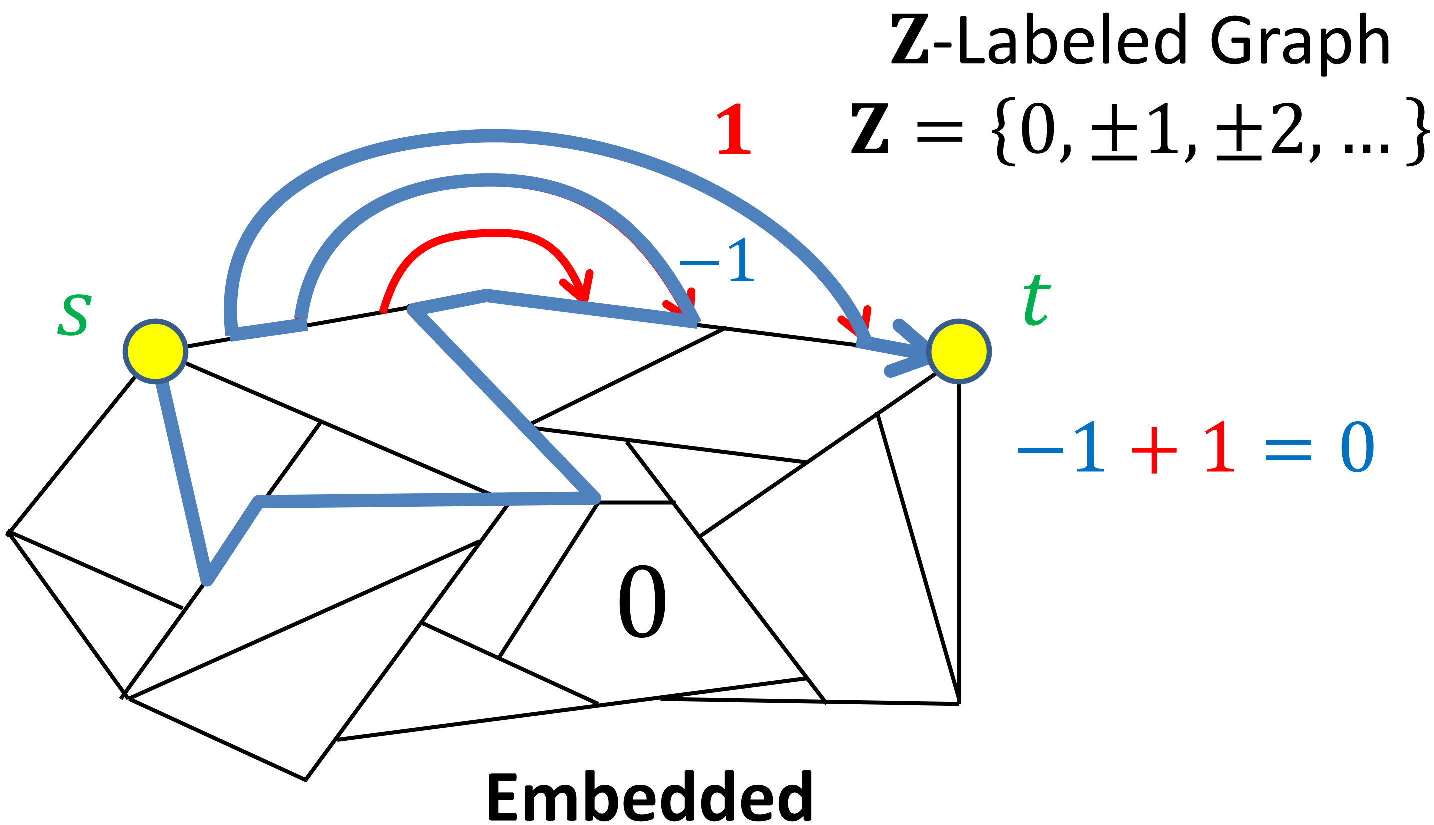


Embedded

$l = \{0, 1\}$

Example 3



$l = \{0, 1\}$

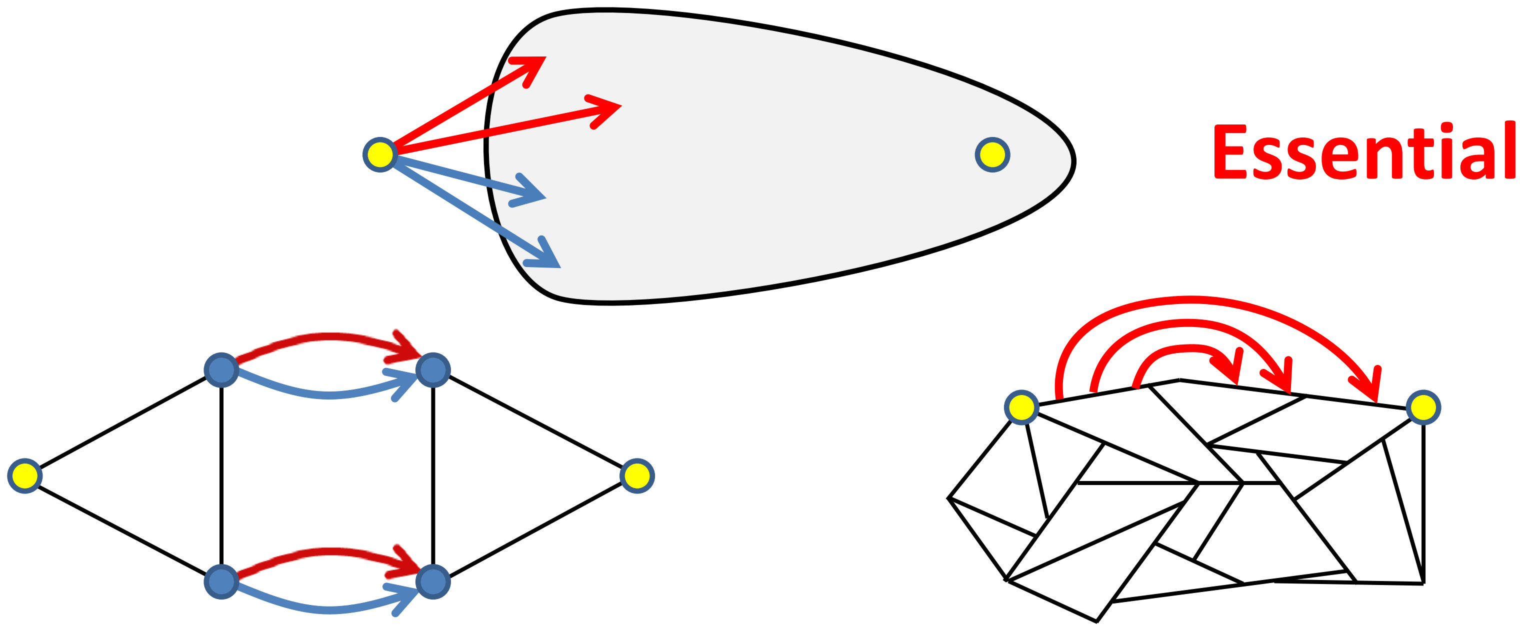
Our Result (Characterization)

Thm. $\underline{l = \{\alpha, \beta\}}$

$\Updownarrow$

Reducible to one of them

[K.–K.–Y. 2015]



Our Result (Characterization)

Thm.

$$l = \{\alpha, \beta\}$$

$$\Updownarrow$$

Reducible to one of them

[K.–K.–Y. 2015]

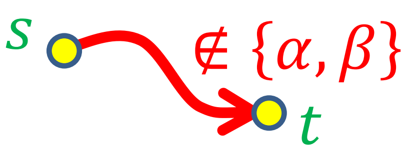
Polytime Testable

- **NOT** Depends on Group
- Assume **Constant-time Group Operations**
  (e.g. Addition, Comparison, …)

Overview

Finding an s–t path **with 2 Labels Forbidden**

Polytime



[K.–K.–Y. 2015]

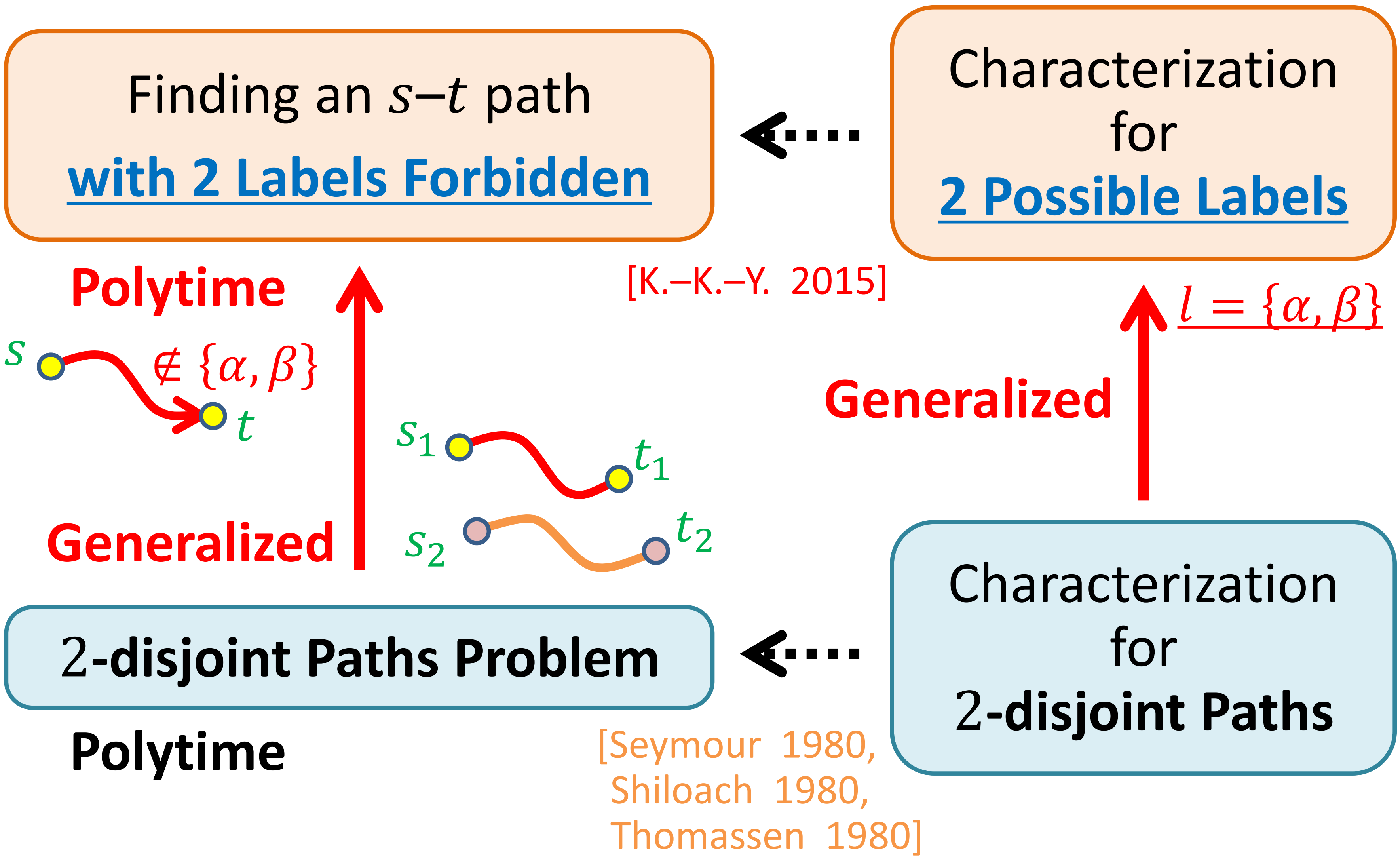
Characterization for **2 Possible Labels**

$l = \{\alpha, \beta\}$

Polytime Testable

- **NOT** Depends on Group
- Assume **Constant-time Group Operations** (e.g. Addition, Comparison, …)

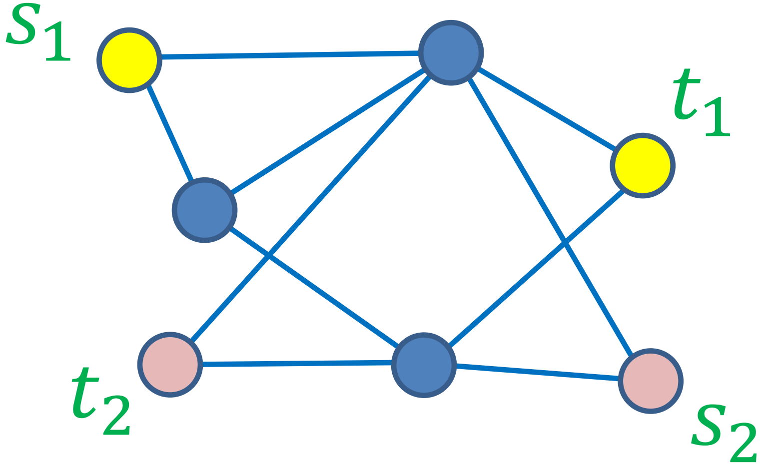
Overview



Finding an s–t path
with 2 Labels Forbidden
Characterization for
2 Possible Labels
[K.–K.–Y. 2015]
Polytime
s
$\notin \{\alpha, \beta\}$
t
$l = \{\alpha, \beta\}$
Generalized
s_1
t_1
s_2
t_2
Generalized
2-disjoint Paths Problem
Characterization for
2-disjoint Paths
Polytime
[Seymour 1980,
Shiloach 1980,
Thomassen 1980]

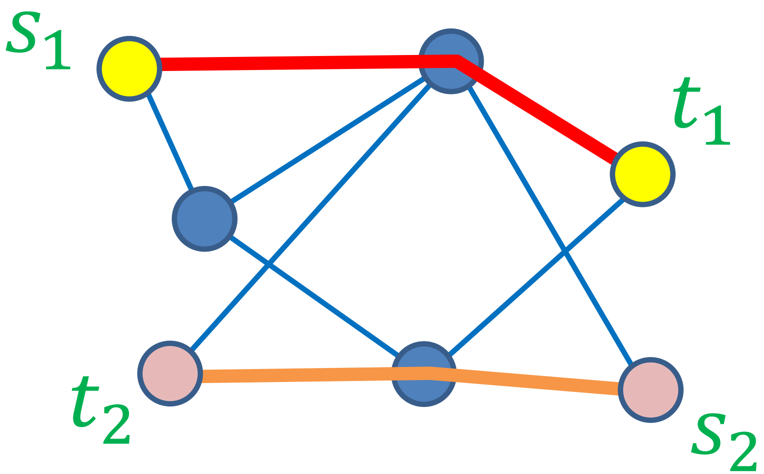
Generalizing 2-disjoint Paths

Undirected Graph



Generalizing 2-disjoint Paths

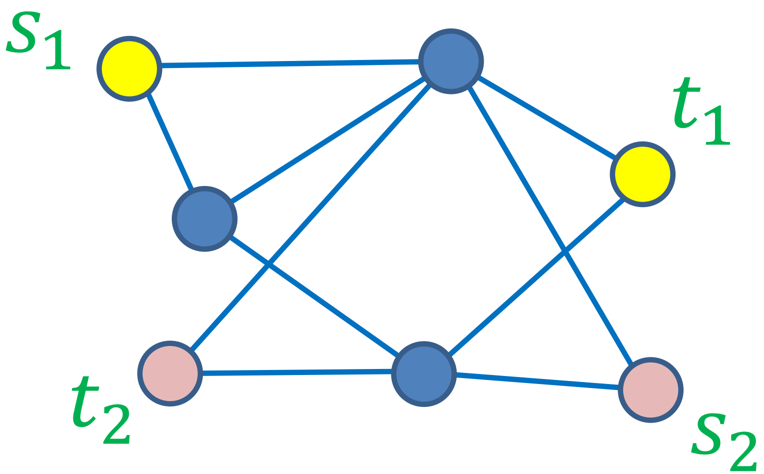
Undirected Graph



2-disjoint paths

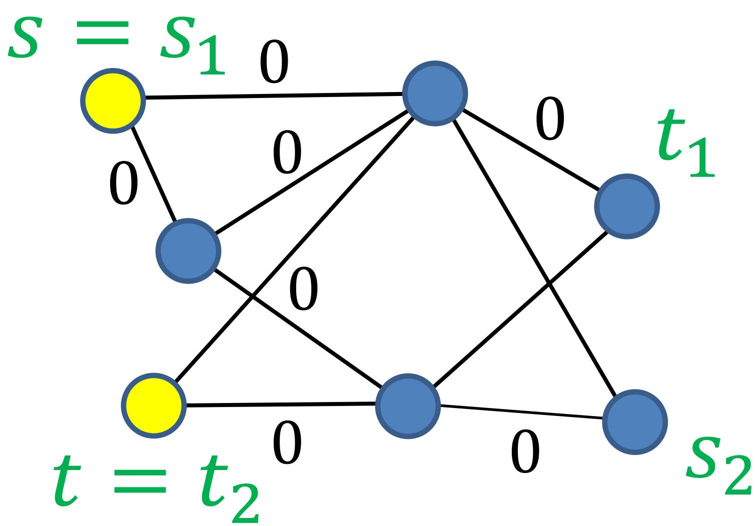
Generalizing 2-disjoint Paths

Undirected Graph



$\mathbf{Z}_3$-Labeled Graph

$\mathbf{Z}_3 = \{-1, 0, 1\}$



Generalizing 2-disjoint Paths

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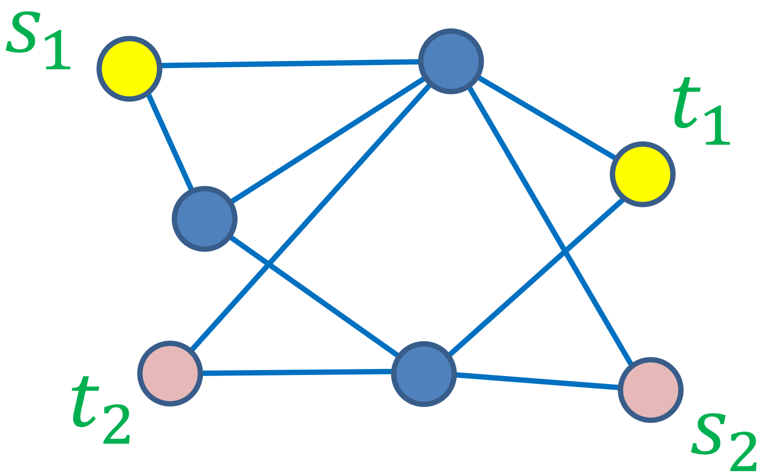

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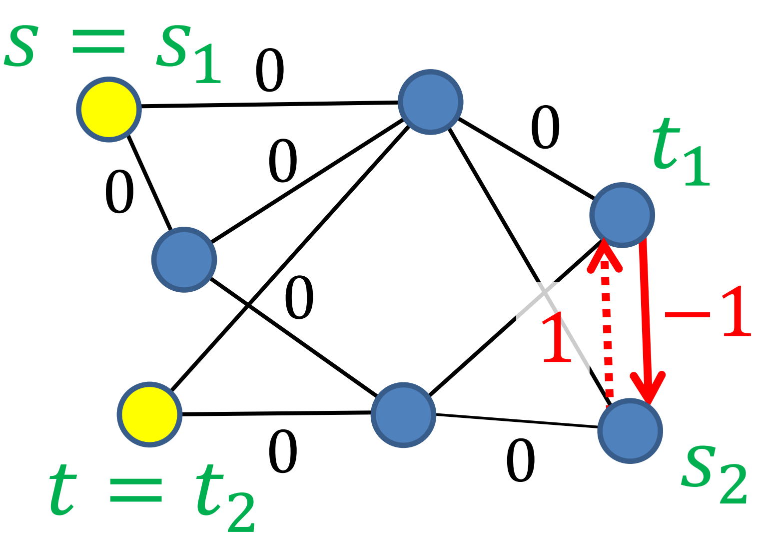
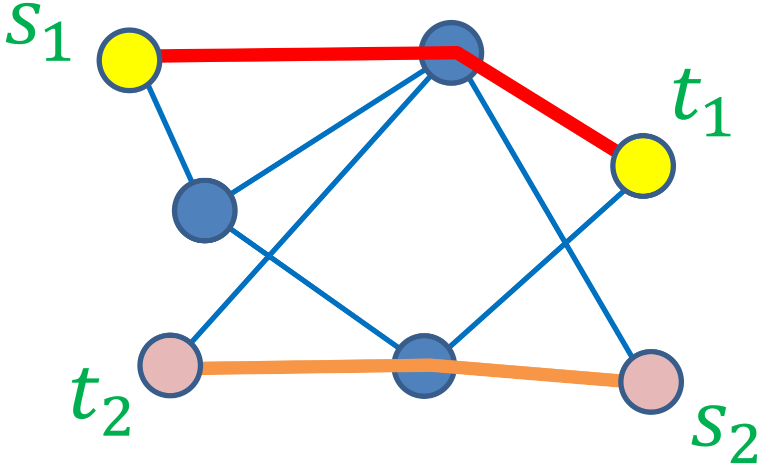


Generalizing 2-disjoint Paths

Undirected Graph



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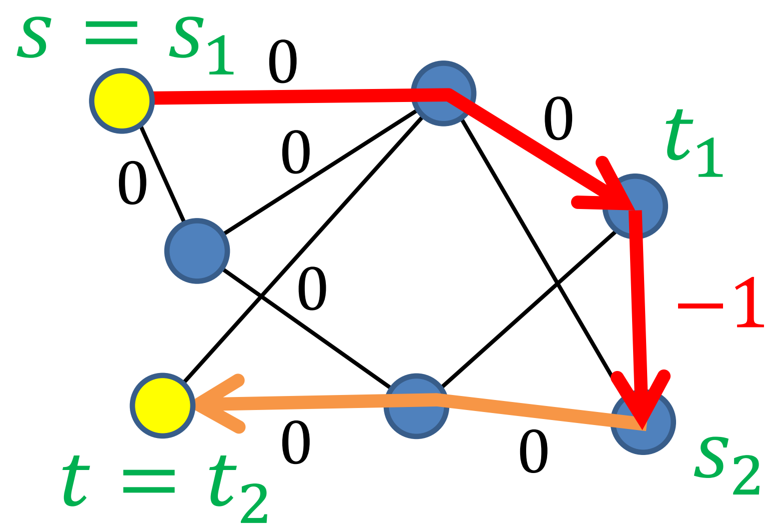

2-disjoint paths $\Leftrightarrow$ Label -1

Generalizing 2-disjoint Paths

Undirected Graph

$\mathbf{Z}_3$-Labeled Graph
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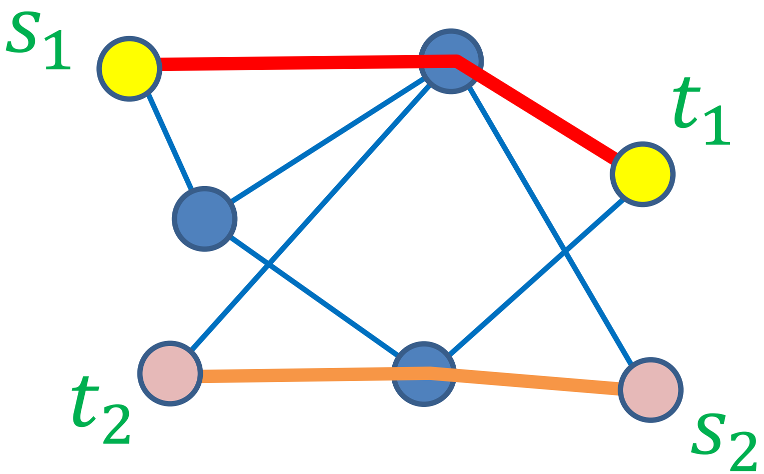
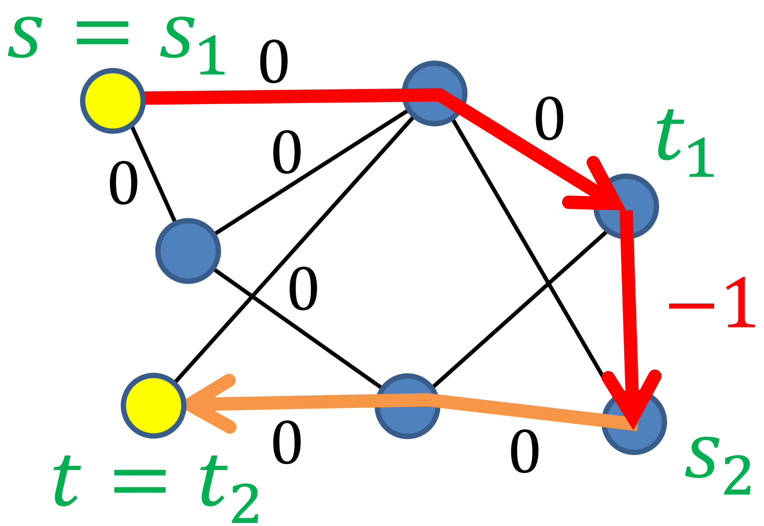




2-disjoint paths $\Leftrightarrow$ Label -1

$\Leftrightarrow$ **Neither 0 nor 1**

Overview

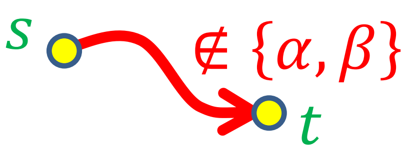
Finding an s–t path
with 2 Labels Forbidden

Characterization
for
2 Possible Labels

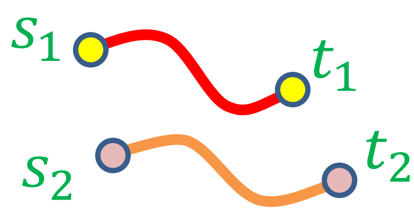
Polytime

[K.–K.–Y. 2015]

$l = \{\alpha, \beta\}$



Generalized



Generalized

2-**disjoint Paths Problem**

Characterization
for
2-**disjoint Paths**

Polytime

[Seymour 1980,
Shiloach 1980,
Thomassen 1980]

Overview

Finding an s–t path
with 2 Labels Forbidden

Polytime

[K.–K.–Y. 2015]

Characterization
for
2 Possible Labels

$l = \{\alpha, \beta\}$

Generalized

Generalized

2-**disjoint Paths Problem**

Polytime

[Seymour 1980,
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Characterization
for
2-**disjoint Paths**

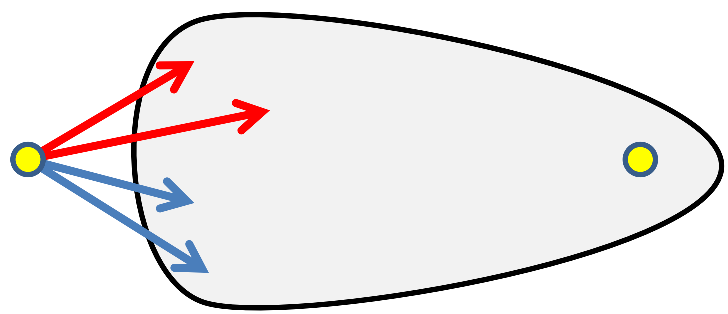
Our Result (Characterization)

Thm. $\underline{l = \{\alpha, \beta\}}$

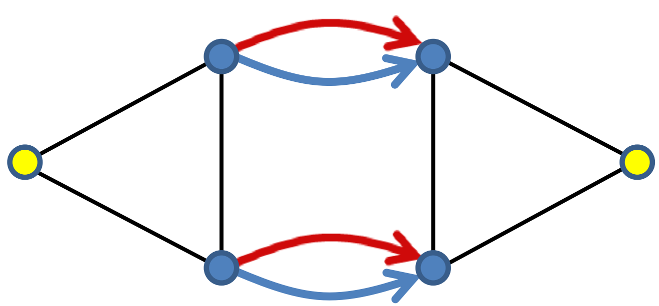
$\Updownarrow$

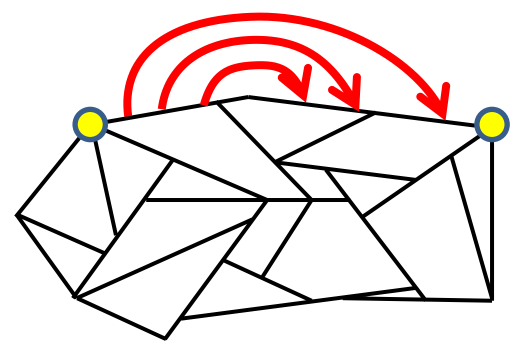
Reducible to one of them

[K.–K.–Y. 2015]



Essential





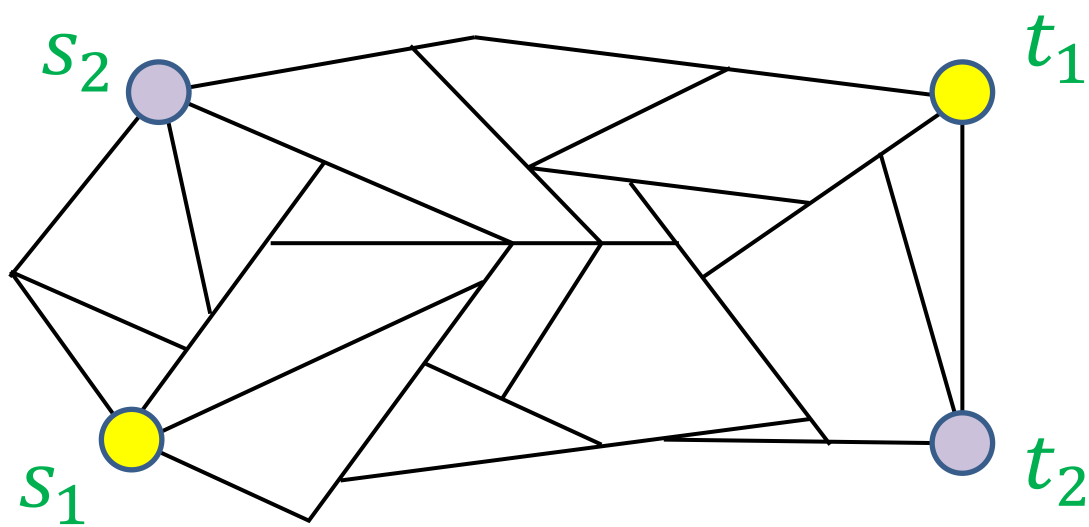
Characterization for 2-disjoint Paths

<u>Thm.</u> **<u>NO</u>** disjoint s_1–t_1, s_2–t_2 paths

⇕

Reducible so that **<u>Planar Embeddable</u>**

[Seymour 1980]



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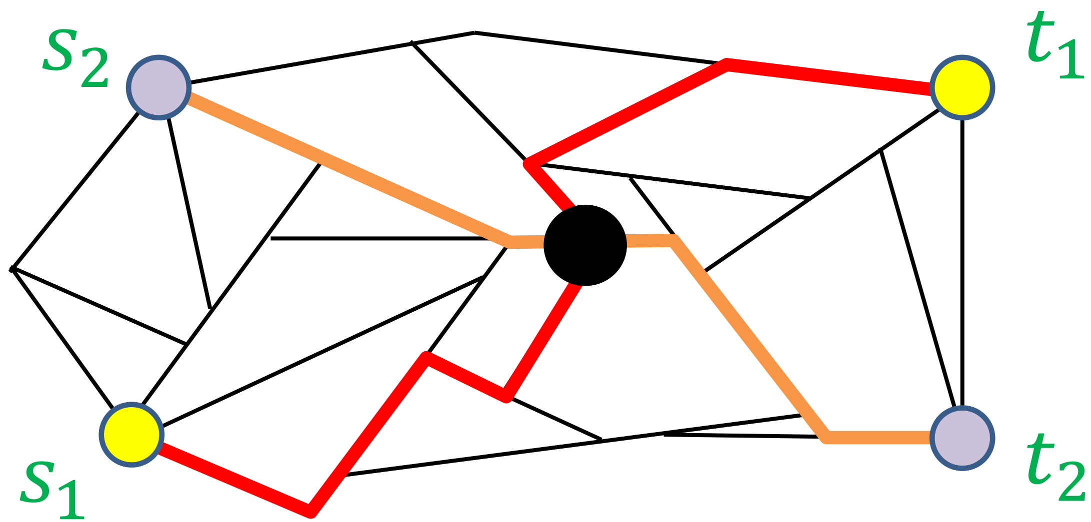
[Seymour 1980]



Essential

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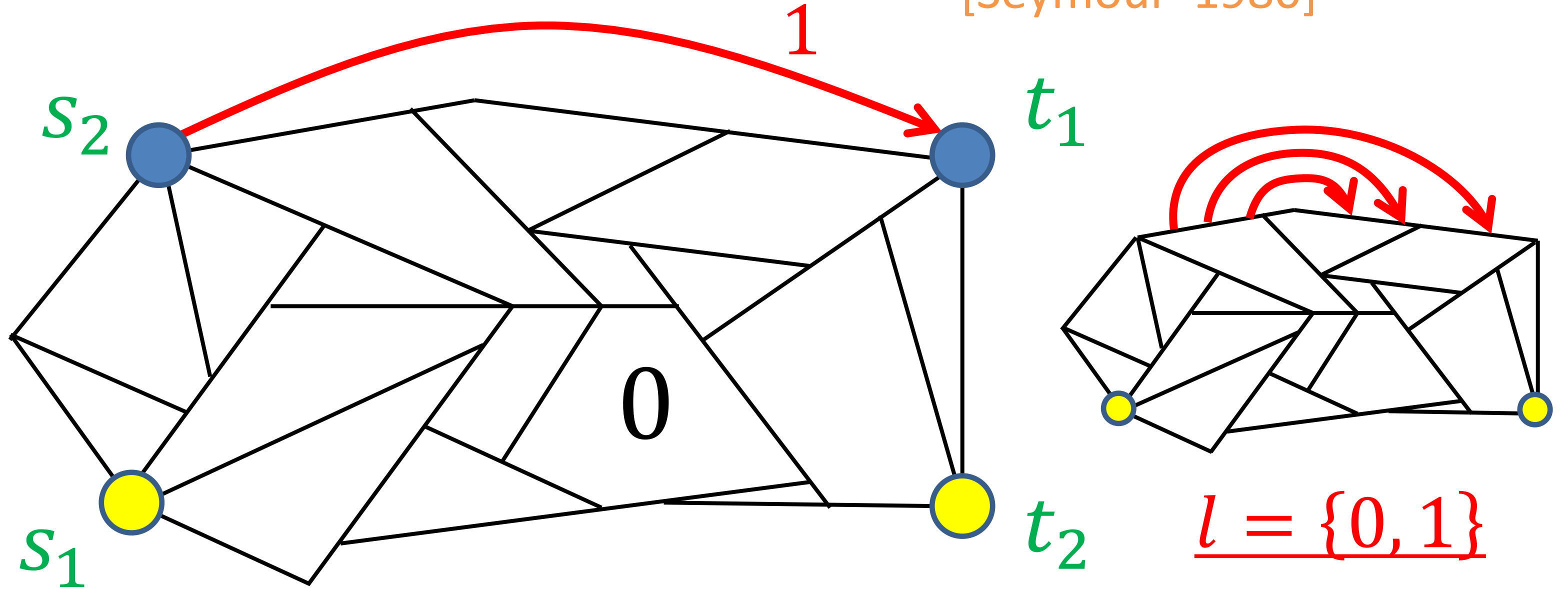
[Seymour 1980]



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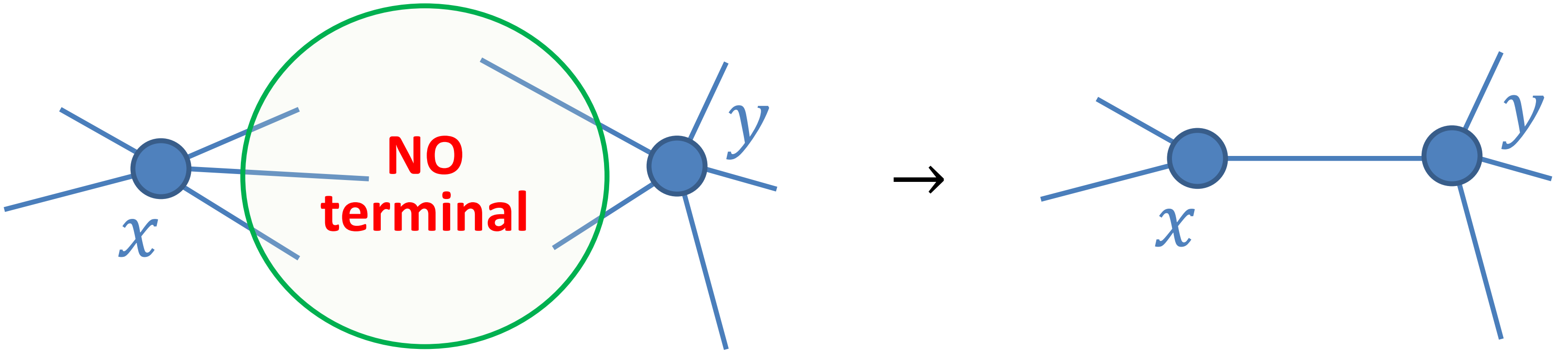
Thm. **NO** disjoint s_1–t_1, s_2–t_2 paths

$$\Updownarrow$$

Reducible so that **Planar Embeddable**

[Seymour 1980]

Contraction of 2-cut



Characterization for 2-disjoint Paths

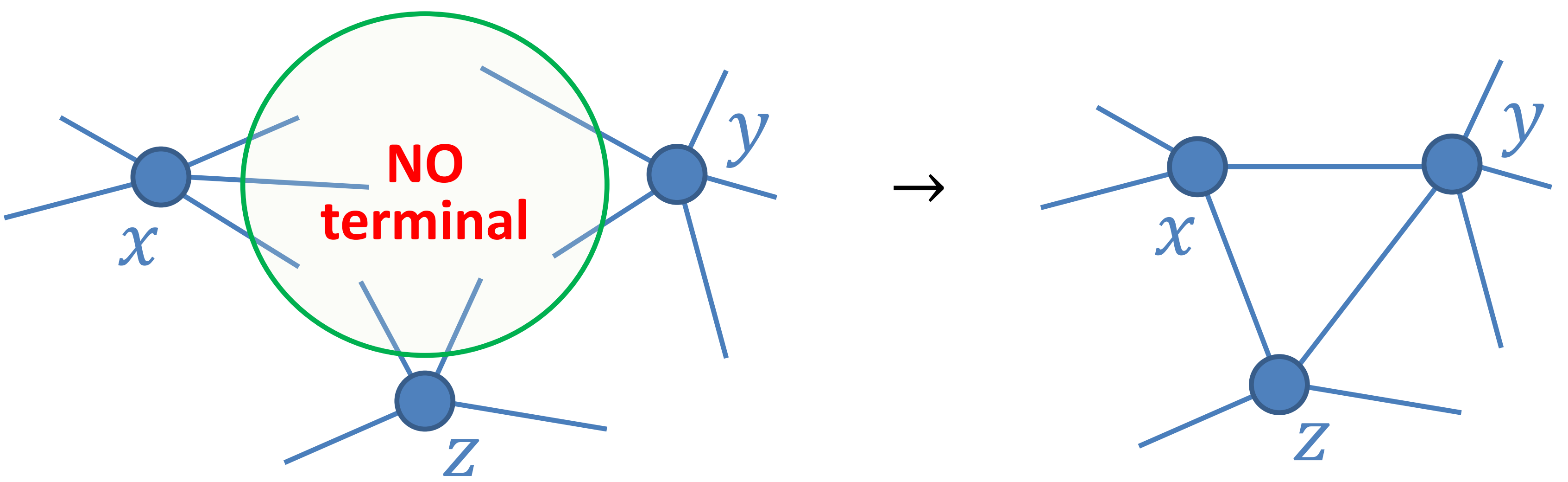
Thm. **NO** disjoint s_1–t_1, s_2–t_2 paths

$$\Updownarrow$$

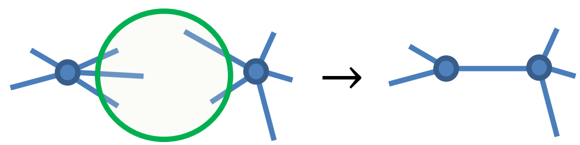
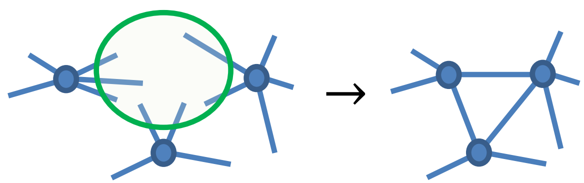
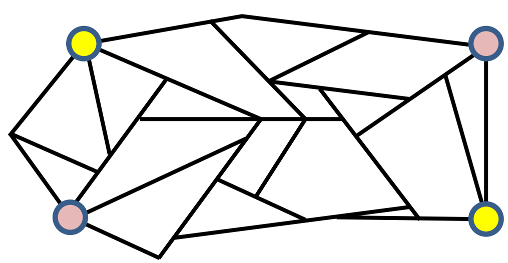
Reducible so that **Planar Embeddable**

[Seymour 1980]

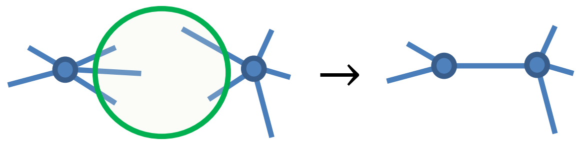
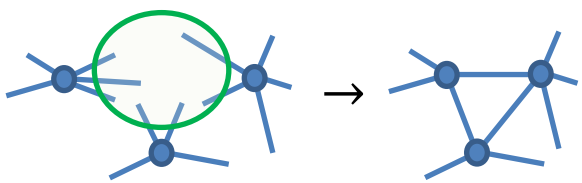
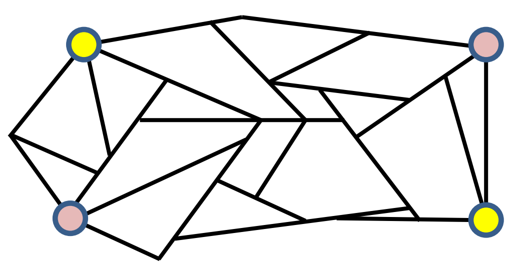
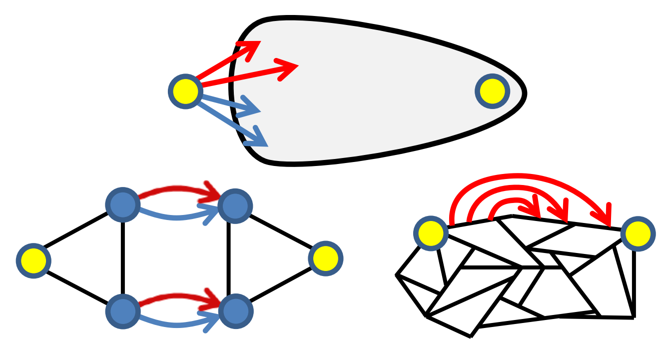
Contraction of 3-cut



Contrast between Two Characterizations

	<u>NO</u> 2-**disjoint Paths** [Seymour 1980]	**Exactly 2 Possible Labels** [K.–K.–Y. 2015]
Reducing Operations	Contraction of 2-cut 	
	Contraction of 3-cut 	
Essential Cases	Planar Embeddable 	

Contrast between Two Characterizations

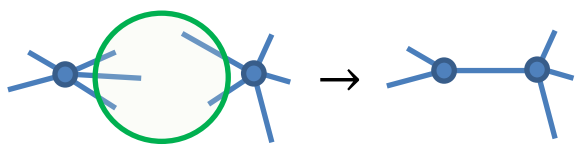
	<u>NO</u> 2-disjoint Paths [Seymour 1980]	**Exactly 2 Possible Labels** [K.–K.–Y. 2015]
Reducing Operations	Contraction of 2-cut 	
	Contraction of 3-cut 	
Essential Cases	Planar Embeddable 	

Contrast between Two Characterizations

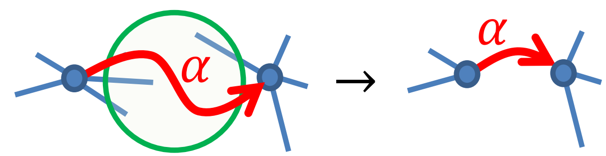
	<u>NO</u> 2-disjoint Paths [Seymour 1980]	**Exactly 2 Possible Labels** [K.–K.–Y. 2015]
Reducing Operations	Contraction of 2-cut	<u>**2-contraction**</u>
	Contraction of 3-cut	<u>**3-contraction**</u>
Essential Cases	Planar Embeddable	

Reducing Operations

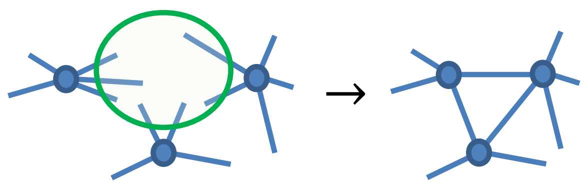
NO 2-disjoint Paths — Contraction of 2-cut



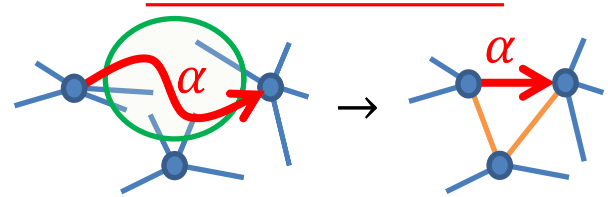
Exactly 2 Possible Labels — 2-contraction



NO 2-disjoint Paths — Contraction of 3-cut

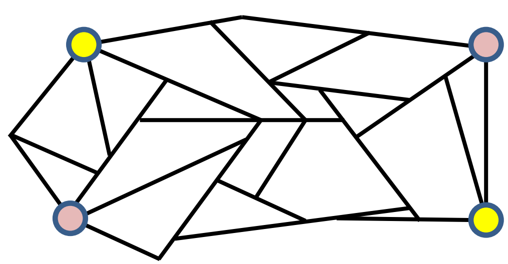


Exactly 2 Possible Labels — 3-contraction

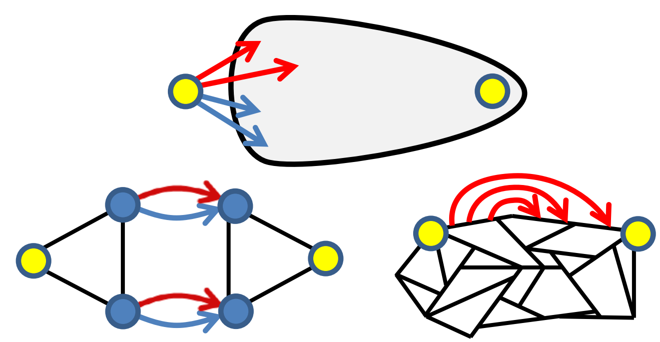


Essential Cases

NO 2-disjoint Paths — Planar Embeddable



Exactly 2 Possible Labels



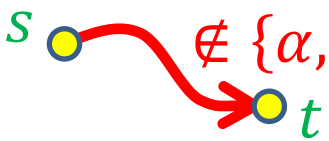
Our Result (Algorithm)

Finding an s–t path

with 2 Labels Forbidden

Characterization for

2 Possible Labels



$\notin \{\alpha, \beta\}$ **Polytime** [K.–K.–Y. 2015]

$l = \{\alpha, \beta\}$

Our Result (Algorithm)

Finding an s–t path
with 2 Labels Forbidden

Characterization
for
2 Possible Labels

$\notin \{\alpha, \beta\}$ **Polytime** [K.–K.–Y. 2015]

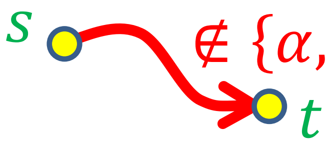
$l = \{\alpha, \beta\}$

- Test "$l \subseteq \{\alpha, \beta\}$ or NOT" (**Based on Our Char.**)

cf. $l = \{\alpha\}$ is Easy

Our Result (Algorithm)

Finding an s–t path **with 2 Labels Forbidden** ⟸ Characterization for **2 Possible Labels**



$\notin \{\alpha, \beta\}$ **Polytime** [K.–K.–Y. 2015]

$l = \{\alpha, \beta\}$

- Test "$l \subseteq \{\alpha, \beta\}$ or NOT" (**Based on Our Char.**)
- $l \subseteq \{\alpha, \beta\}$ $\rightarrow$ **Certification for "NO"**

Our Result (Algorithm)

Finding an s–t path
with 2 Labels Forbidden

Characterization
for
2 Possible Labels

$\notin \{\alpha, \beta\}$ **Polytime** [K.–K.–Y. 2015]

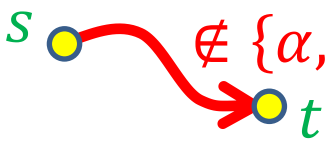
$l = \{\alpha, \beta\}$

- Test "$l \subseteq \{\alpha, \beta\}$ or NOT" (**Based on Our Char.**)
- $l \subseteq \{\alpha, \beta\}$ → **Certification for "NO"**
- $l \nsubseteq \{\alpha, \beta\}$ → $\exists \gamma \in l \setminus \{\alpha, \beta\}$

Our Result (Algorithm)

Finding an s–t path **with 2 Labels Forbidden**

⟵ Characterization for **2 Possible Labels**



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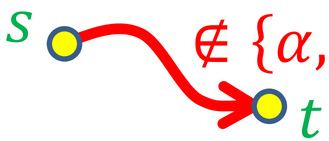
- Test "$l \subseteq \{\alpha, \beta\}$ or NOT" (**Based on Our Char.**)
- $l \subseteq \{\alpha, \beta\}$ $\rightarrow$ **Certification for "NO"**
- $l \nsubseteq \{\alpha, \beta\}$ $\rightarrow$ $\exists \gamma \in l \setminus \{\alpha, \beta\}$
  - $|l| \leq 2$ $\rightarrow$ Paths of **ALL Possible Labels**
  - $|l| \geq 3$ $\rightarrow$ Paths of **3 Distinct Labels**

Our Result (Algorithm)

Finding an s–t path
with 2 Labels Forbidden

$\Leftarrow\cdots$

Characterization
for
2 Possible Labels



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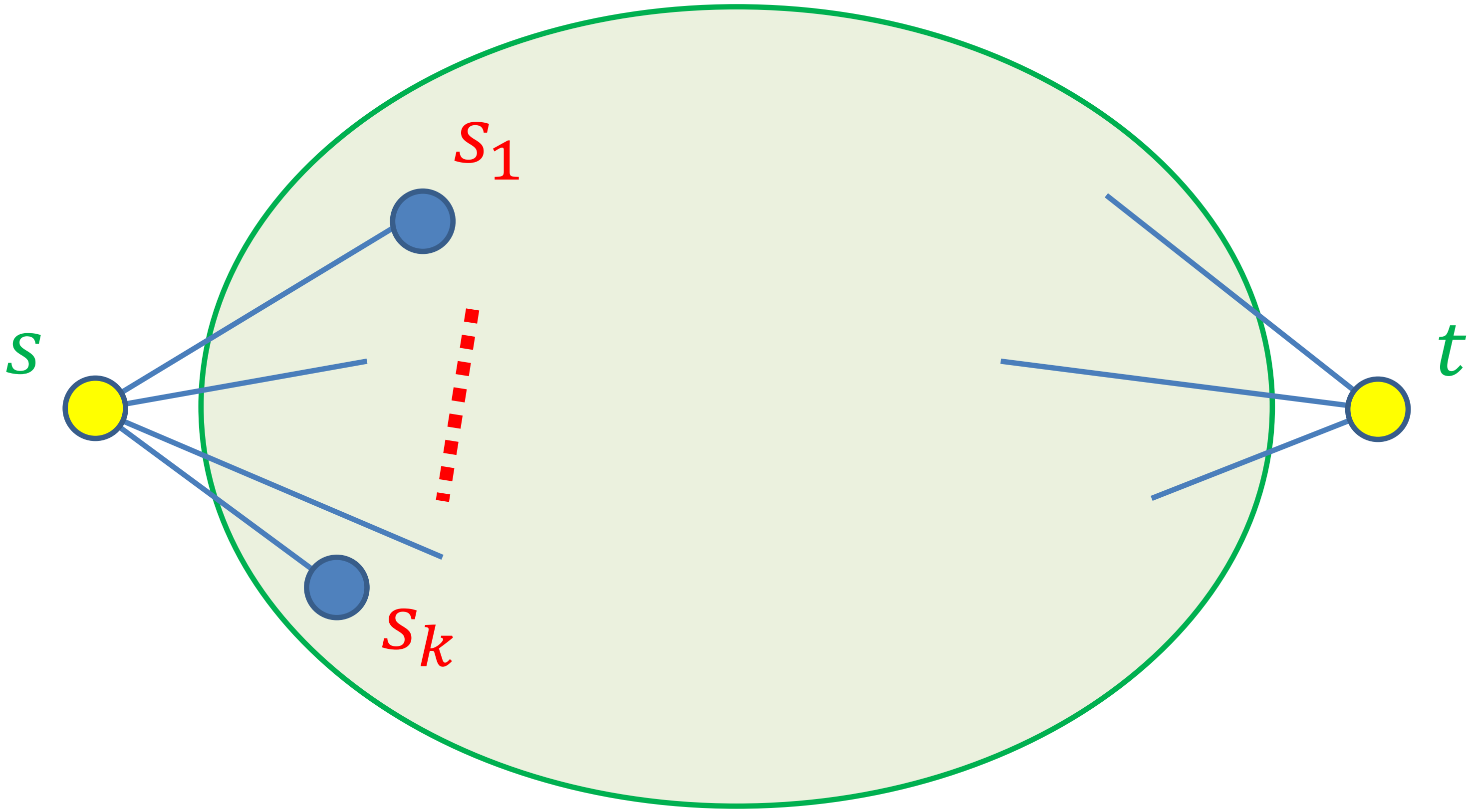
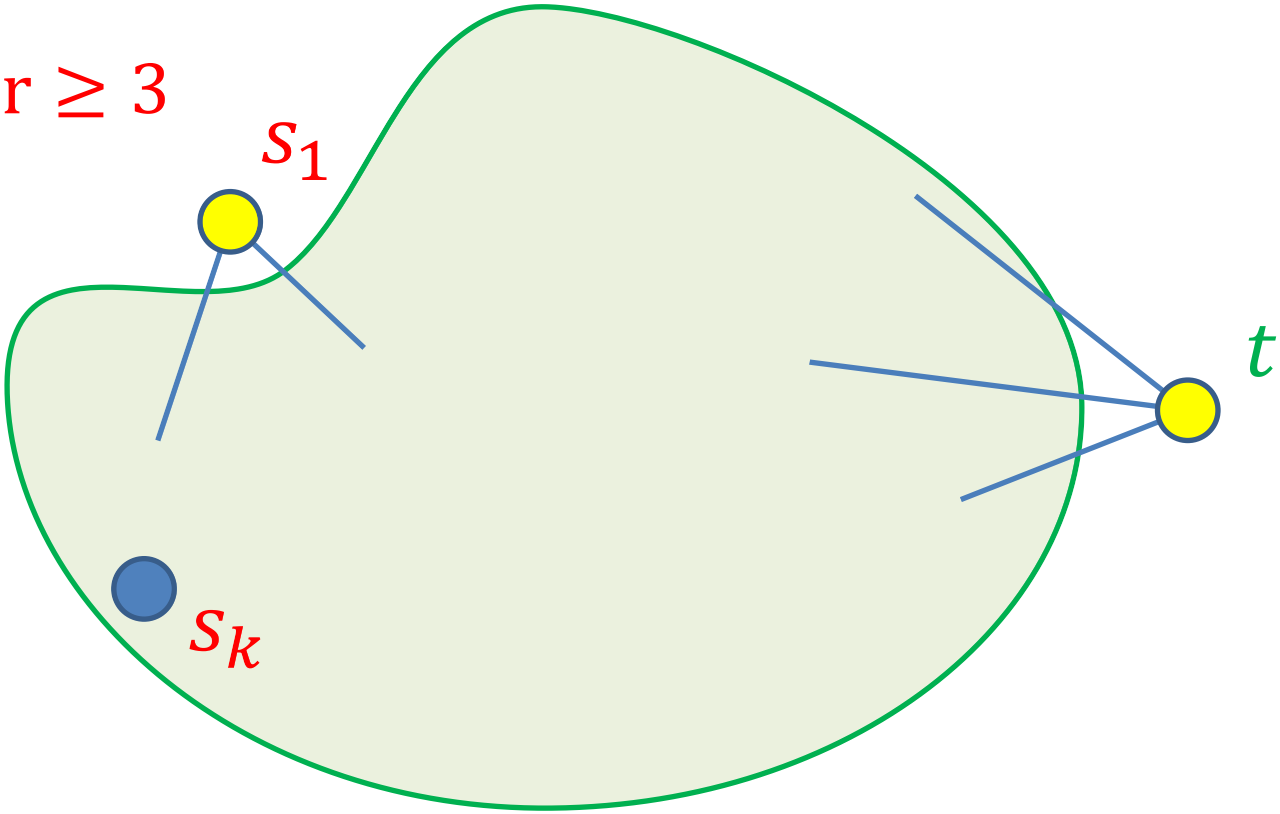
  - $|l| \geq 3 \rightarrow$ Paths of **3 Distinct Labels**

Finding s–t Paths of 3 Distinct Labels

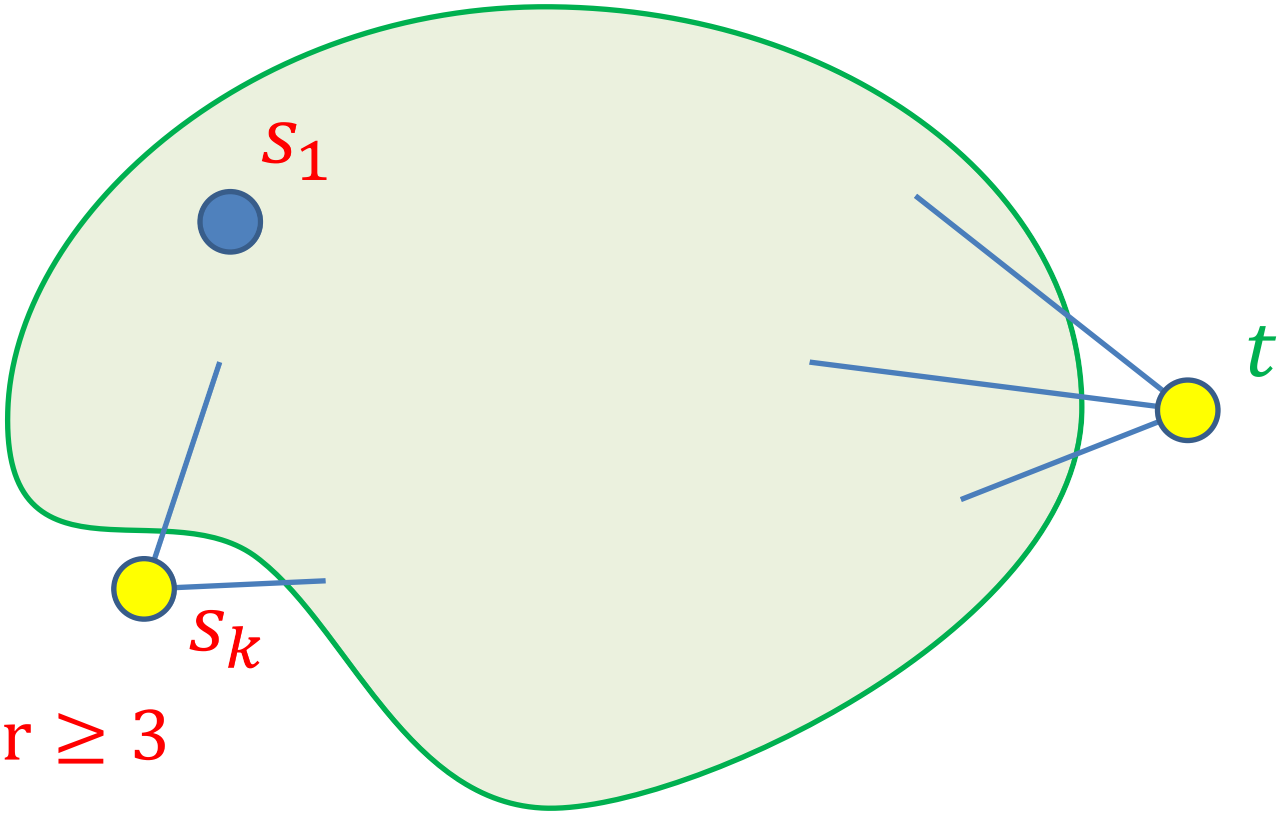


Finding s–t Paths of 3 Distinct Labels

$|l| \leq 2$ or ≥ 3

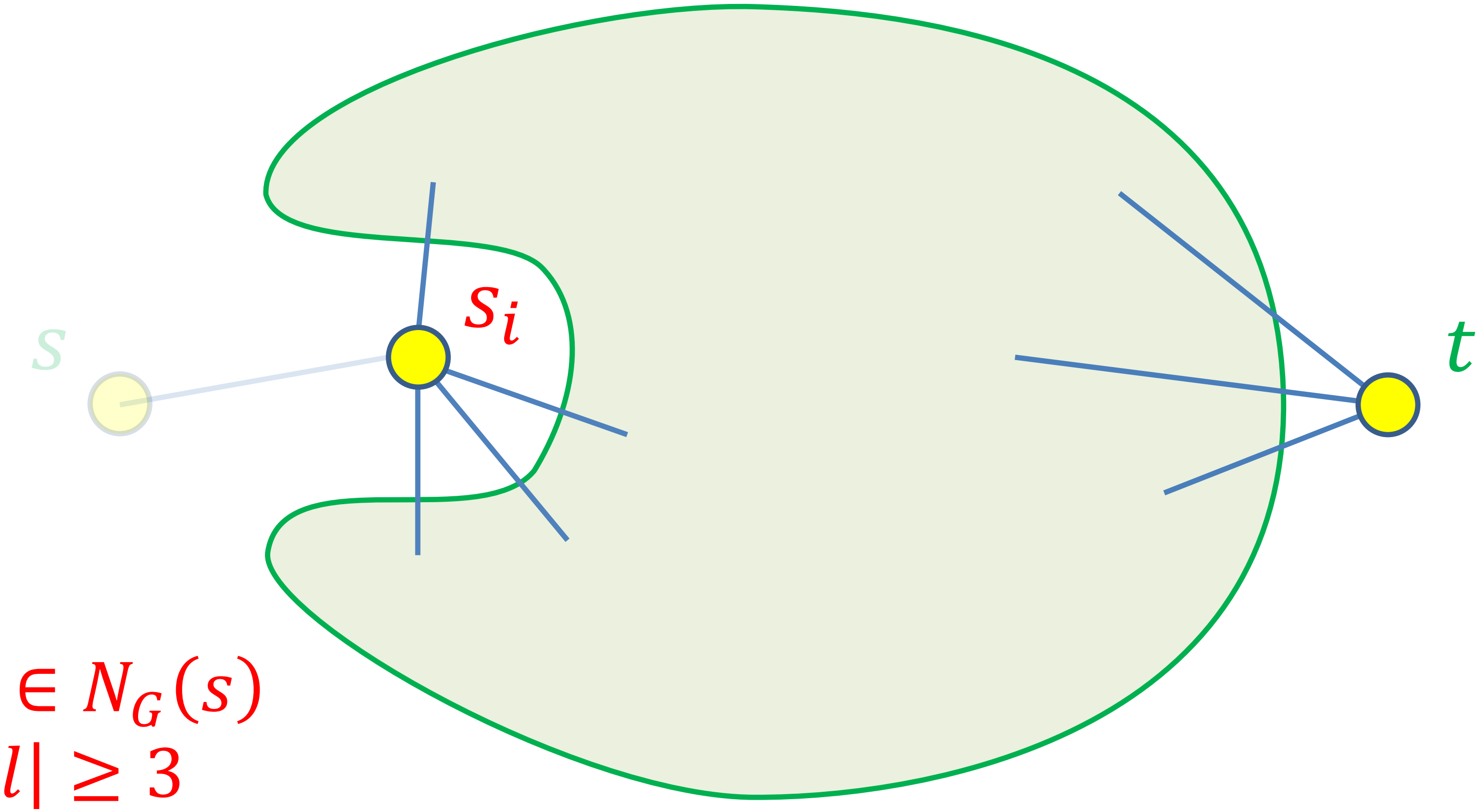


Finding s–t Paths of 3 Distinct Labels



$|l| \leq 2$ or ≥ 3

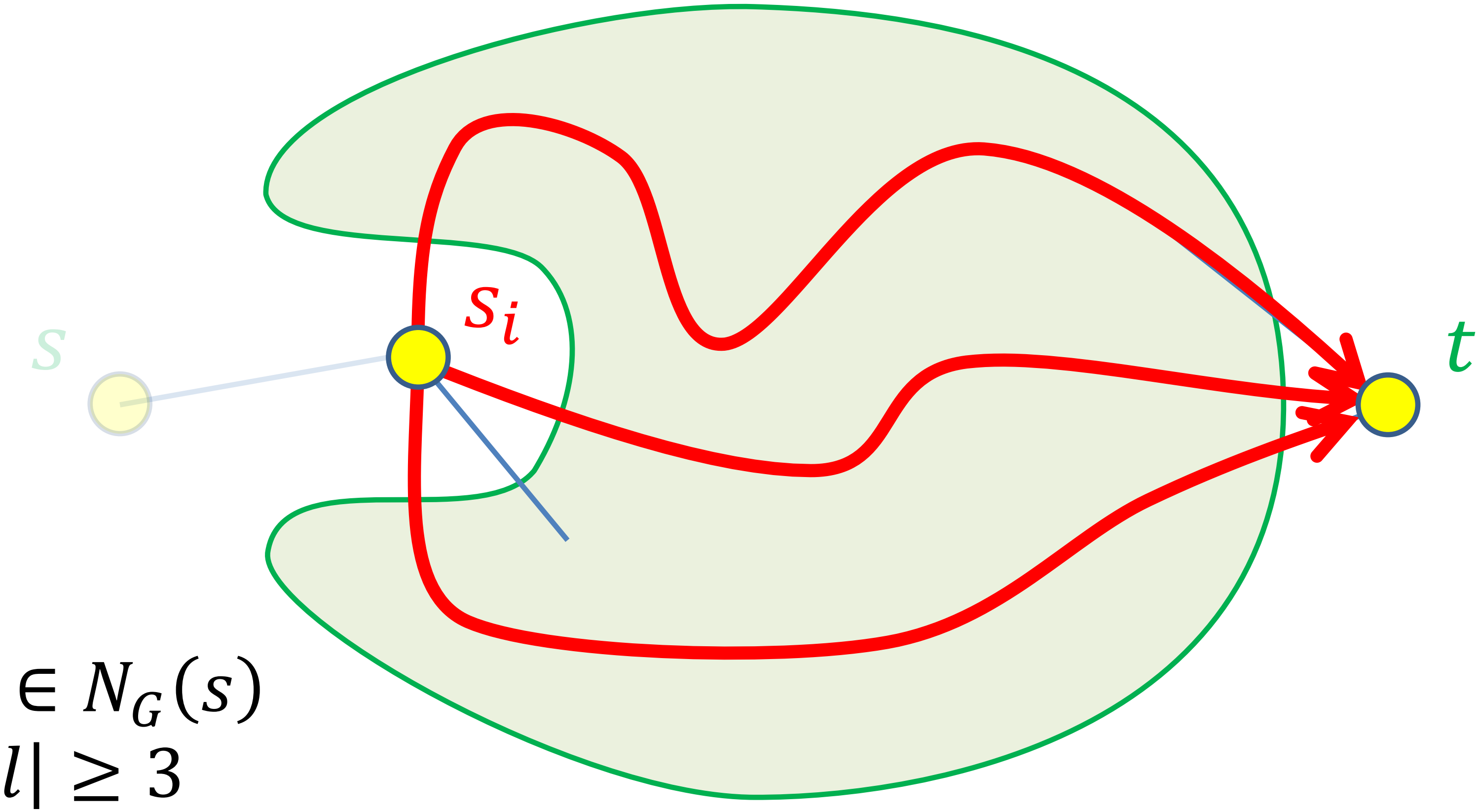
Finding s–t Paths of 3 Distinct Labels



$\exists s_i \in N_G(s)$

$|l| \geq 3$

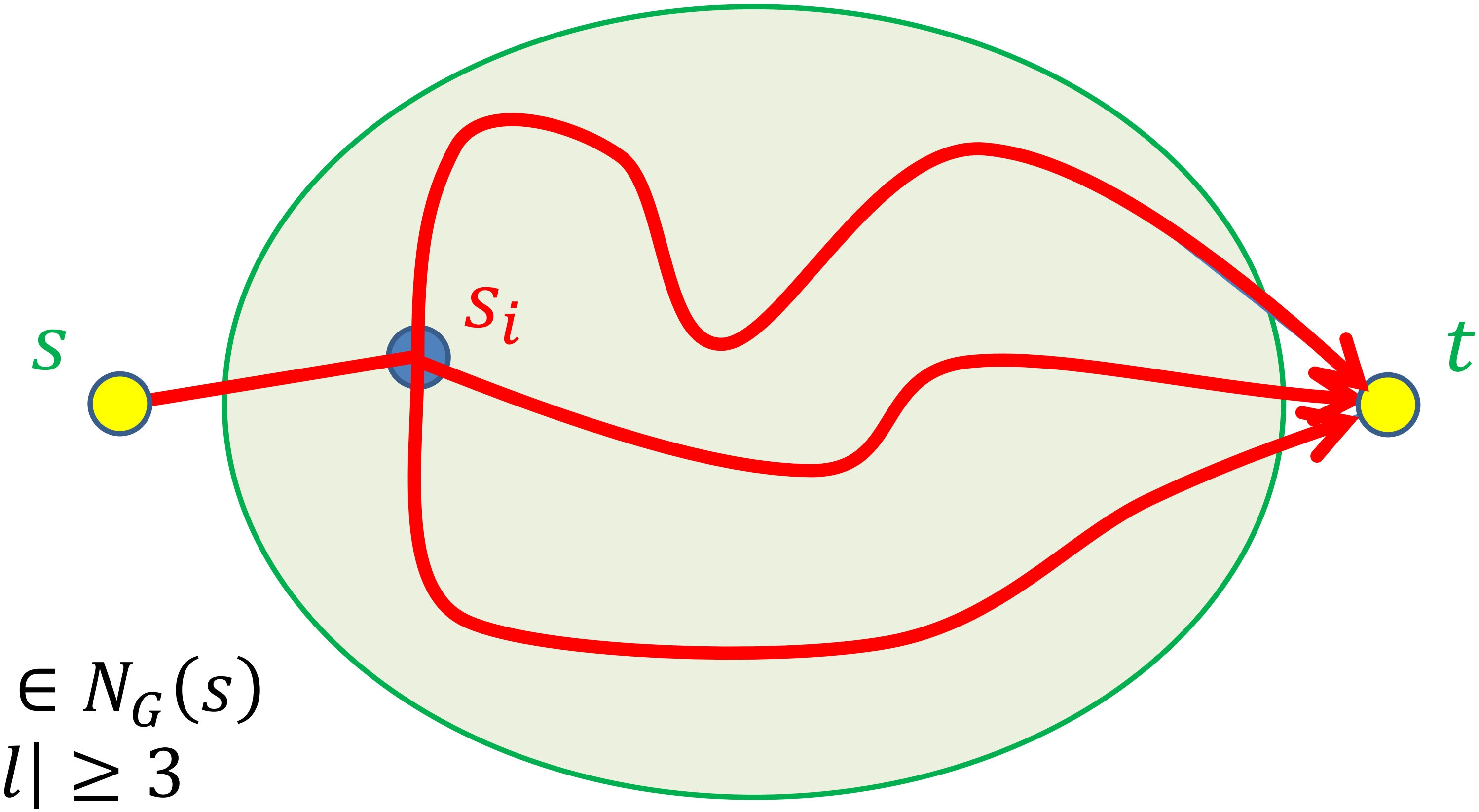
Finding s–t Paths of 3 Distinct Labels



$\exists s_i \in N_G(s)$
$|l| \geq 3$

Find 3 s_i–t paths **Recursively**

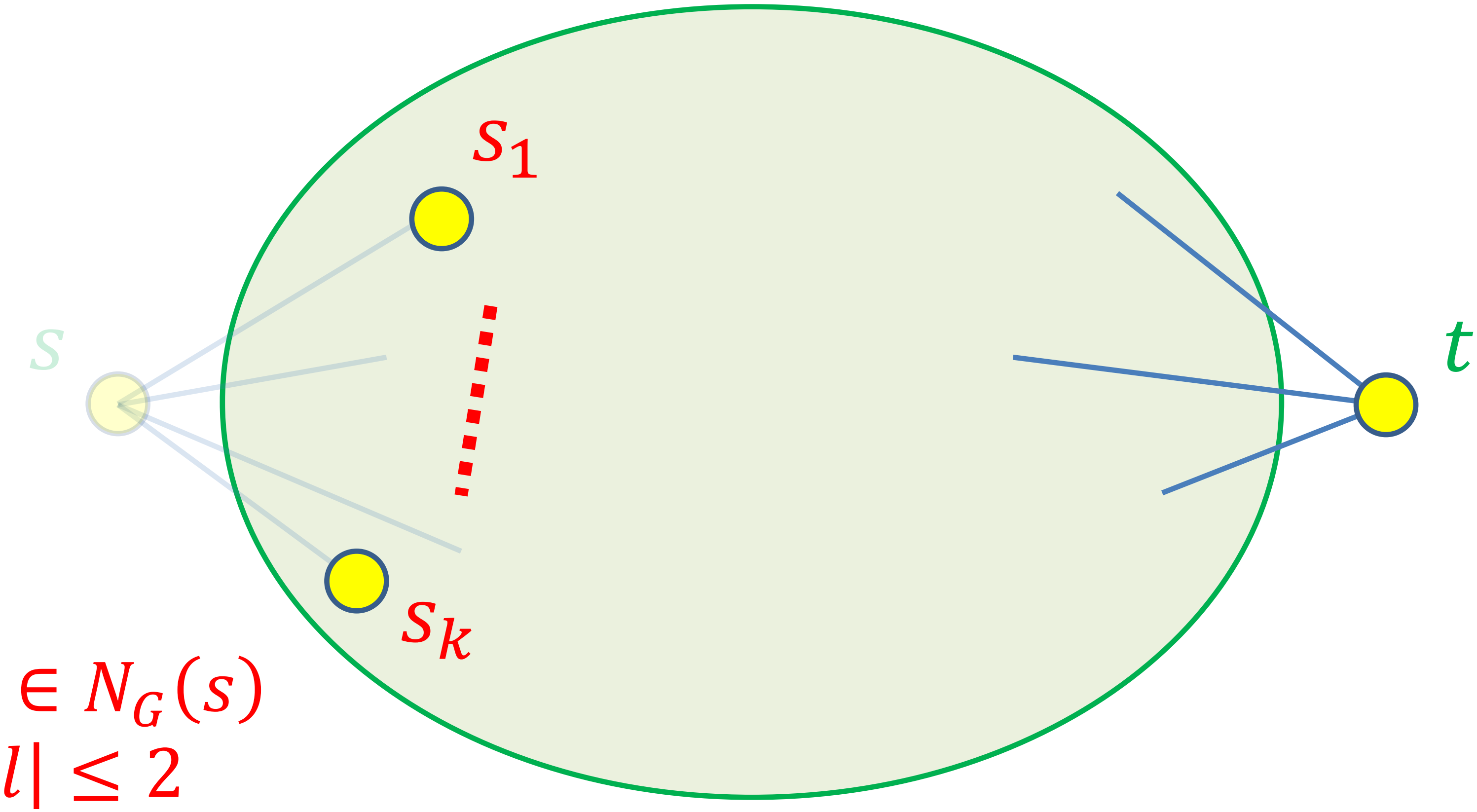
Finding s–t Paths of 3 Distinct Labels



$\exists s_i \in N_G(s)$
$|l| \geq 3$

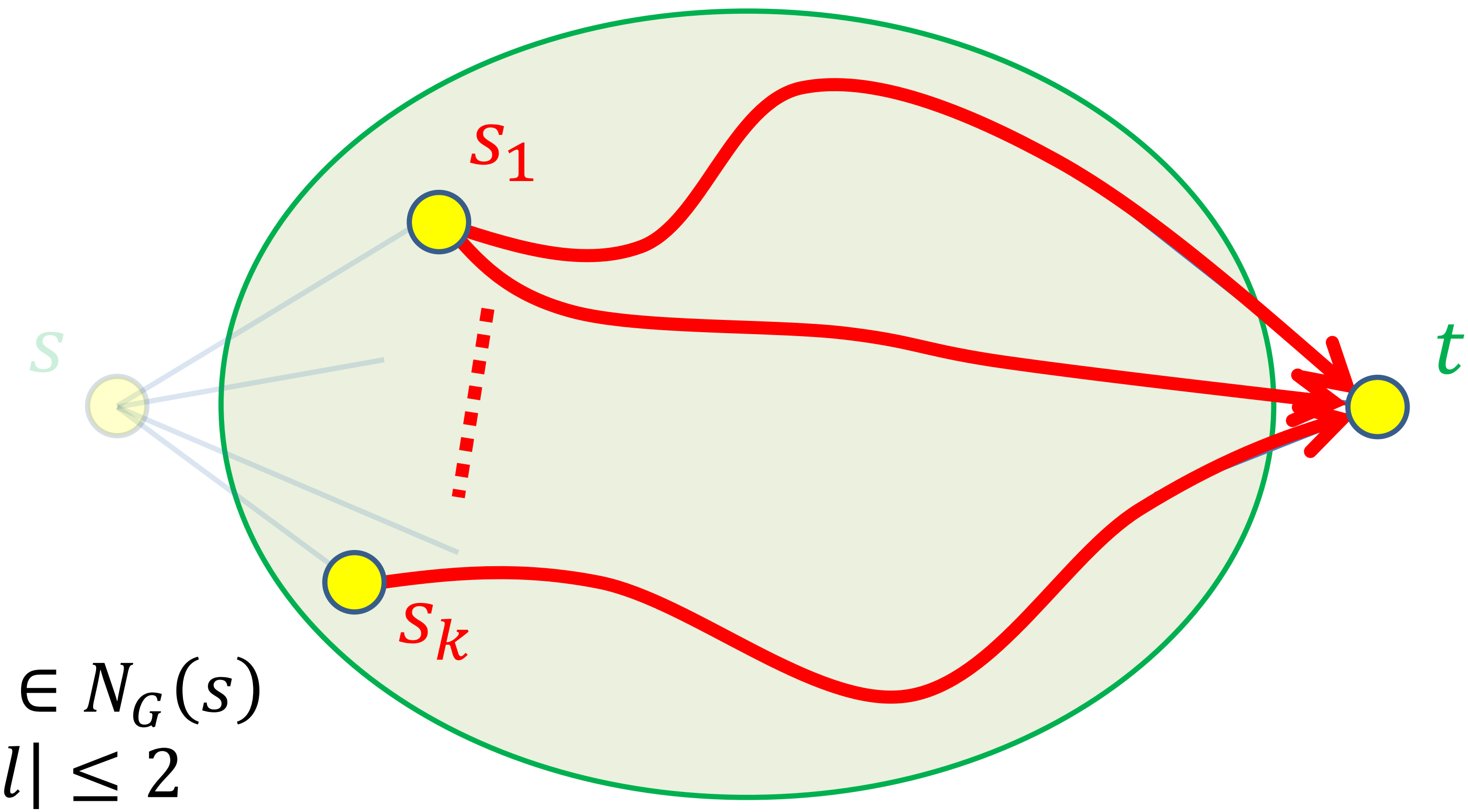
Find 3 s_i–t paths Recursively, and **Extend** them

Finding s–t Paths of 3 Distinct Labels



$\forall s_i \in N_G(s)$
$|l| \leq 2$

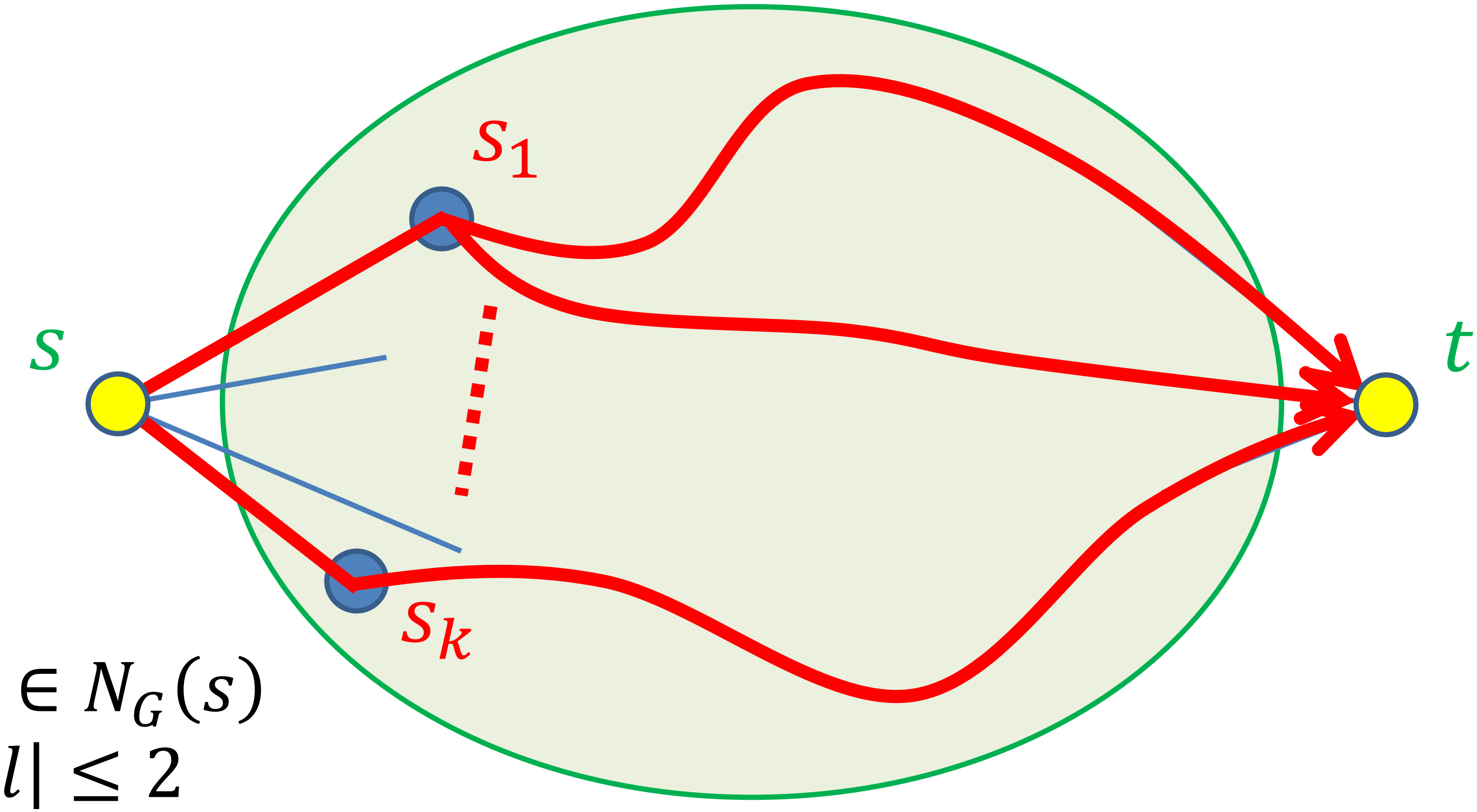
Finding s–t Paths of 3 Distinct Labels



$\forall s_i \in N_G(s)$
$|l| \leq 2$

Get s_i–t paths of **ALL** possible labels

Finding s–t Paths of 3 Distinct Labels



$\forall s_i \in N_G(s)$
$|l| \leq 2$

Get s_i–t paths of ALL possible labels, **Extend** and **Select**

Conclusion

- **Characterization** for a Group-Labeled Graph with **Exactly 2 Possible Labels** of s–t paths
  - **Polytime Testable**
  - Extends Char. for 2**-disjoint Paths** [Seymour 1980]
- **Algorithm** to find an s–t path **with 2 Labels Forbidden**
  - **Polytime**
  - **NOT** Depends on Group

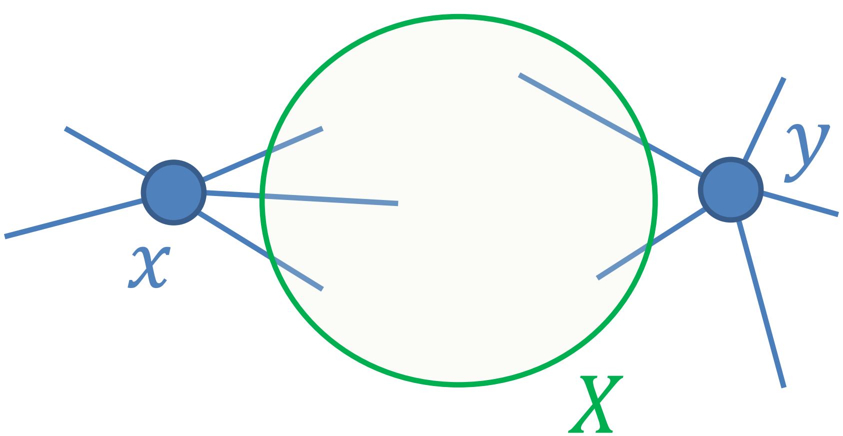
    Non-abelian or **Infinite** is OK
    If Group Operations in Const. time

2-contraction

2-contraction of $X \subseteq V \setminus \{s, t\}$ with $N_G(X) = \{x, y\}$

$\overset{\text{def}}{\Updownarrow}$

- Remove all vertices in X
- Add an edge from x to y with each label of an x–y path through X



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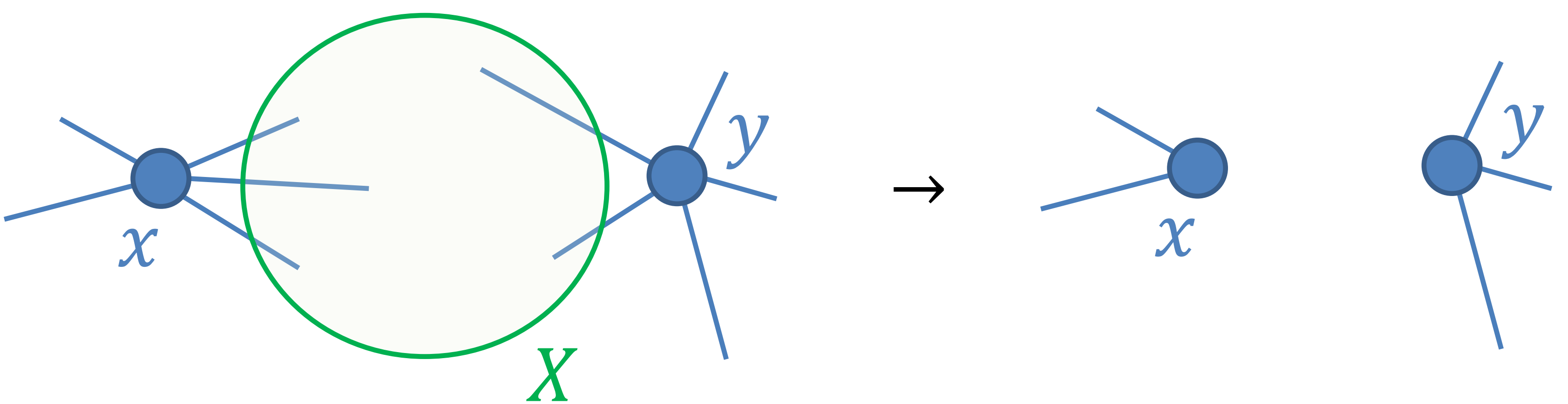

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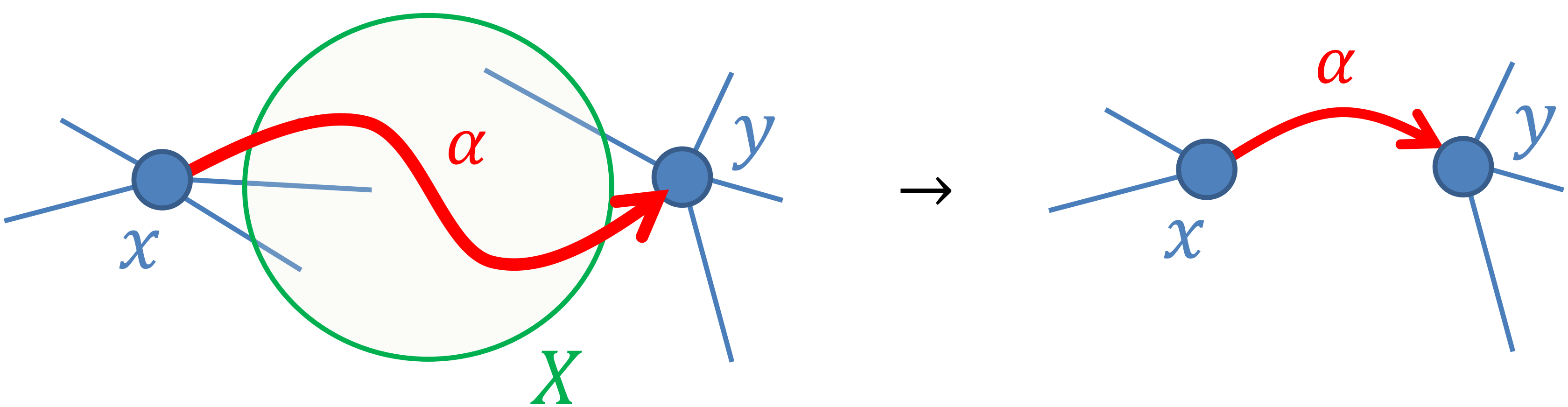

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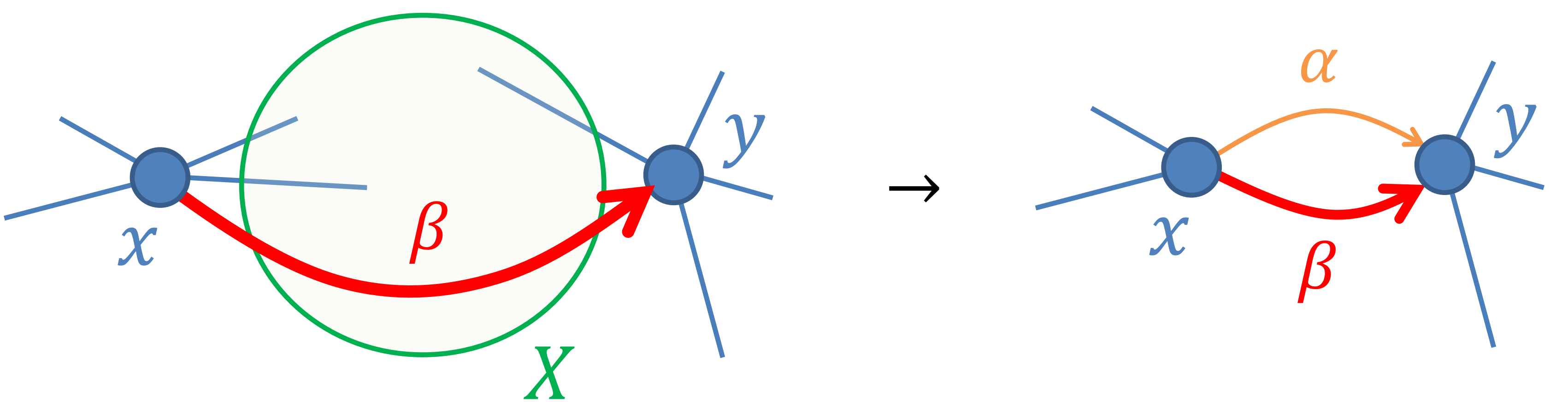
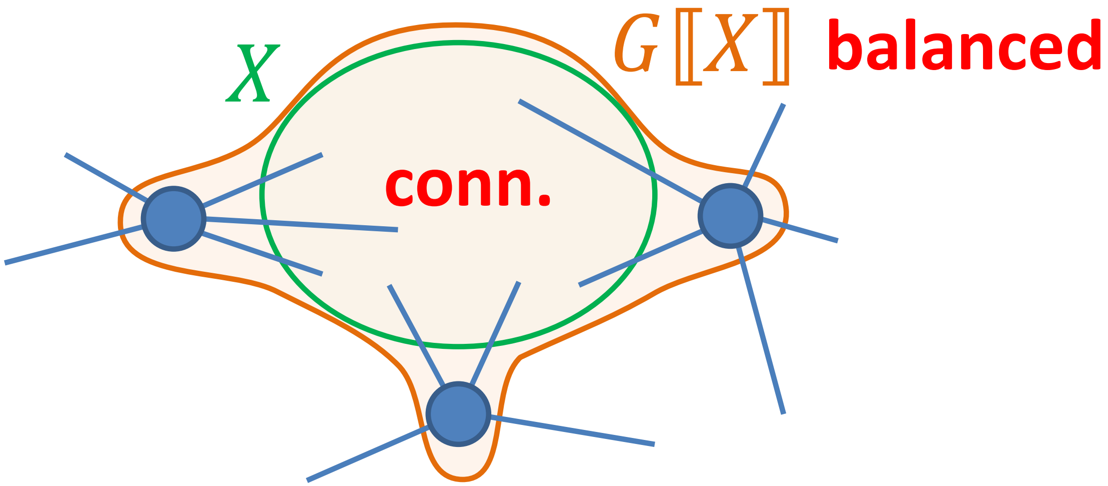


3-contraction

3-contraction of $X \subseteq V \setminus \{s, t\}$ with $|N_G(X)| = 3$
($G[X]$ is **connected** and $G[\![X]\!]$ is **balanced**)

def $\Updownarrow$

- Remove all vertices in X
- Add an edge xy w. label $l(G[\![X]\!]; x, y)$ $\left(\forall x, y \in N_G(X)\right)$



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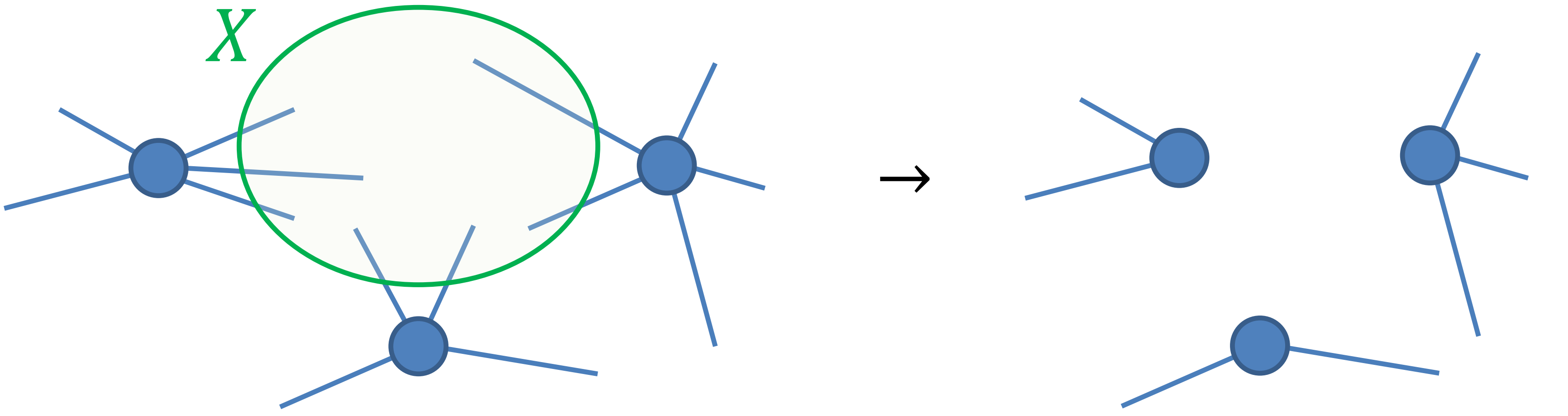

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