Shortest Disjoint $\mathcal{S}$-paths via Weighted Linear Matroid Parity

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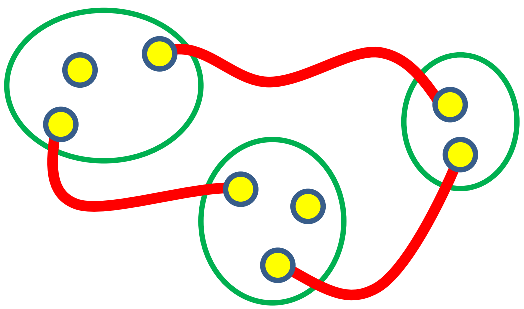
Osaka University, Japan

ISAAC 2016 @Sydney December 12, 2016

Overview

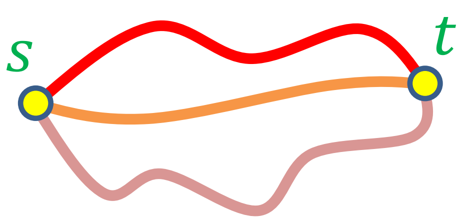
Find **Maximum-Cardinality** Feasible Solution

P



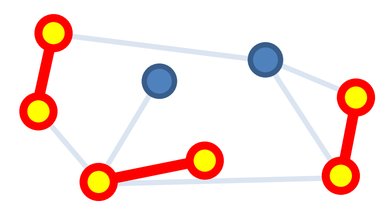
Disjoint $\mathcal{S}$-paths

Polytime
via Linear Matroid Parity
[Lovász 1980, 1981]
[Schrijver 2003]

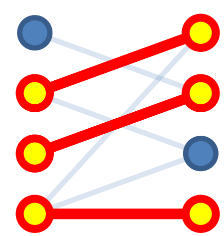


Disjoint
s–t Paths

Non-bipartite
Matching

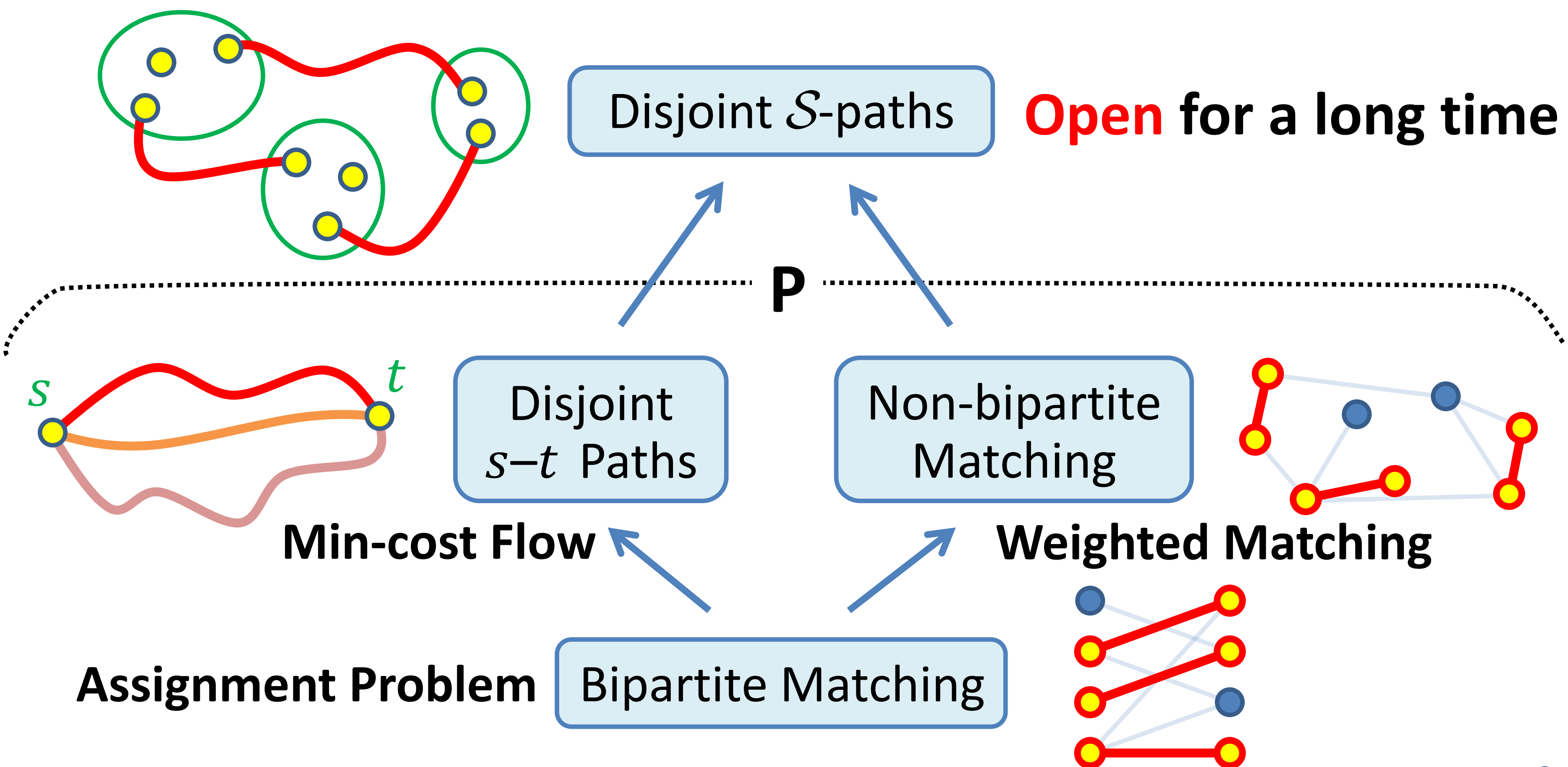


Bipartite Matching



Overview

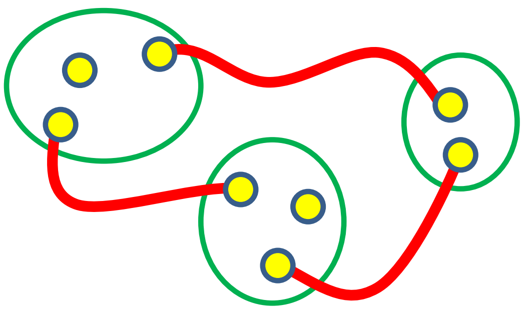
Find **Minimum-Cost** **Fixed-Cardinality** Feasible Solution



Overview

<u>Find</u> **<u>Minimum-Cost</u>** **<u>Fixed-Cardinality</u>** Feasible Solution

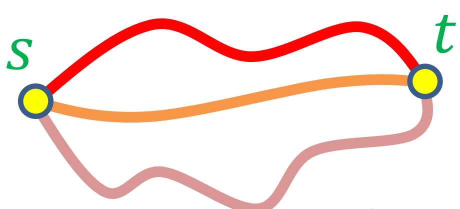
P



Disjoint $\mathcal{S}$-paths

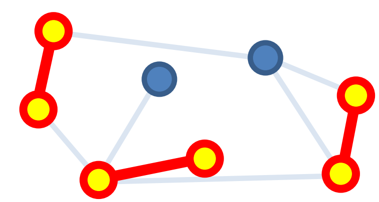
~~Open for a long time~~

Polytime
via Weighted L.M.P.



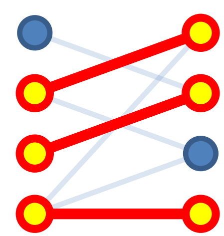
Disjoint
s–t Paths

Non-bipartite
Matching



Min-cost Flow

Weighted Matching



Assignment Problem

Bipartite Matching

Successful Scenario

	Disjoint $\mathcal{S}$-paths	Linear Matroid Parity
Cardinality ver.	Polytime [Lovász 1980] [Schrijver 2003]	Polytime e.g., [Lovász 1981] [Gabow–Stallmann 1986] [Cheung–Lau–Leung 2016]
Weighted ver.	Polytime??	Polytime?

Reduce

Reduce?

Successful Scenario

	Disjoint $\mathcal{S}$-paths		Linear Matroid Parity
Cardinality ver.	Polytime [Lovász 1980] [Schrijver 2003]	**Reduce** $\longrightarrow$	Polytime e.g., [Lovász 1981] [Gabow–Stallmann 1986] [Cheung–Lau–Leung 2016]
Weighted ver.	Polytime??	Reduce? $\dashrightarrow$	**Polytime!** Announced by [Iwata 2013][Pap 2013]

Successful Scenario

	Disjoint $\mathcal{S}$-paths		Linear Matroid Parity
Cardinality ver.	Polytime [Lovász 1980] [Schrijver 2003]	**Reduce** $\rightarrow$	Polytime e.g., [Lovász 1981] [Gabow–Stallmann 1986] [Cheung–Lau–Leung 2016]
Weighted ver.	**Polytime!** [This Talk]	**Reduce!** $\rightarrow$	Polytime Announced by [Iwata 2013][Pap 2013]

Outline

- Preliminaries
  - Disjoint $\mathcal{S}$-paths
  - Linear Matroid Parity
  - Reduction in Cardinality Case
- Result
  - How to Extend to Weighted Case
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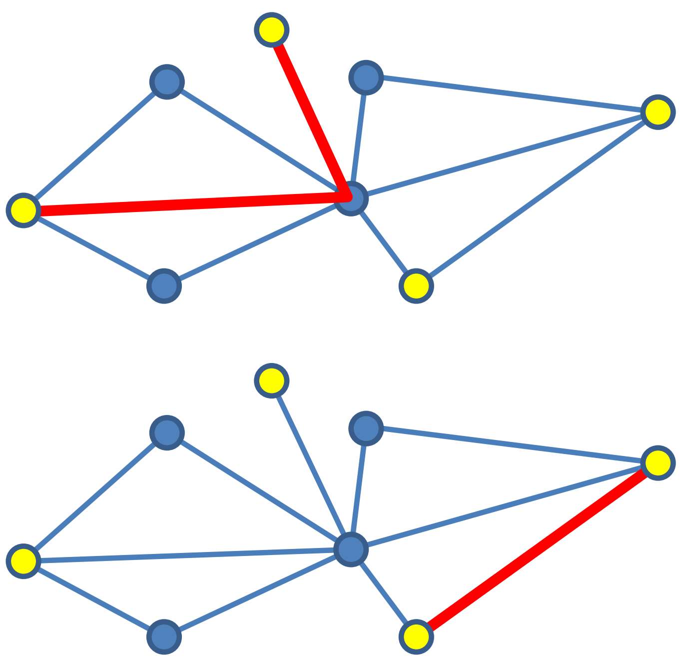
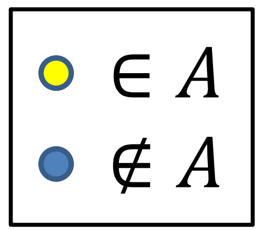
A-paths and $\mathcal{S}$-paths

$G = (V, E)$: Undirected Graph

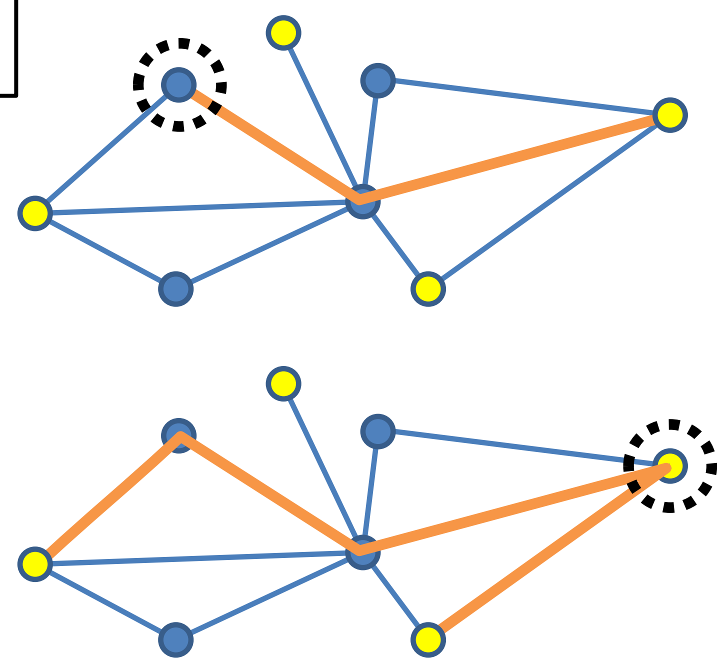
$A \subseteq V$: Terminal Set

A-paths





NOT A-paths



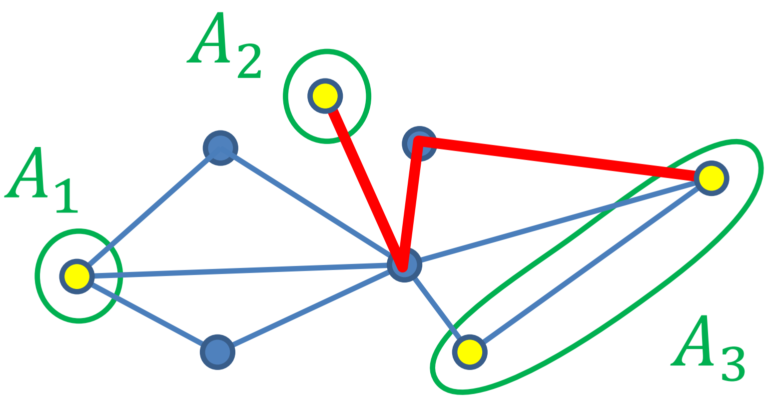
A-paths and $\mathcal{S}$-paths

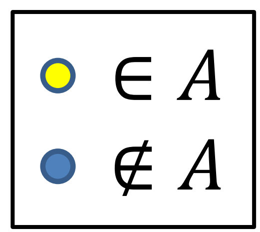
$G = (V, E)$: Undirected Graph

$A \subseteq V$: Terminal Set

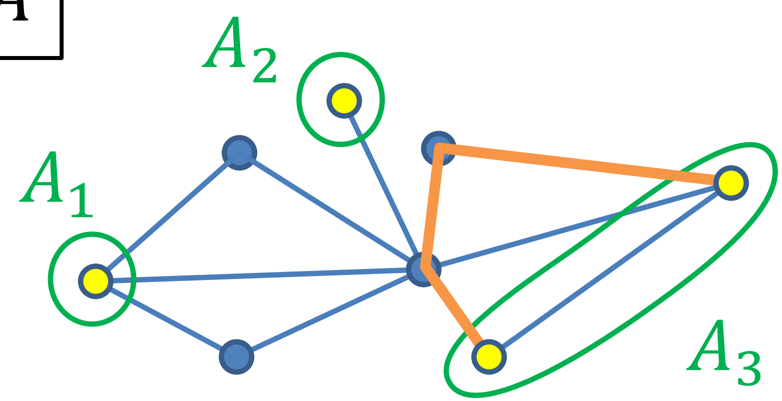
$\mathcal{S} = \{A_1, A_2, \dots, A_k\}$: **Partition** of A

$\mathcal{S}$-path





NOT $\mathcal{S}$-path



Disjoint $\mathcal{S}$-paths Problem

Given $G = (V, E)$: Undirected Graph

$A \subseteq V$: Terminal Set, $\mathcal{S}$: Partition of A

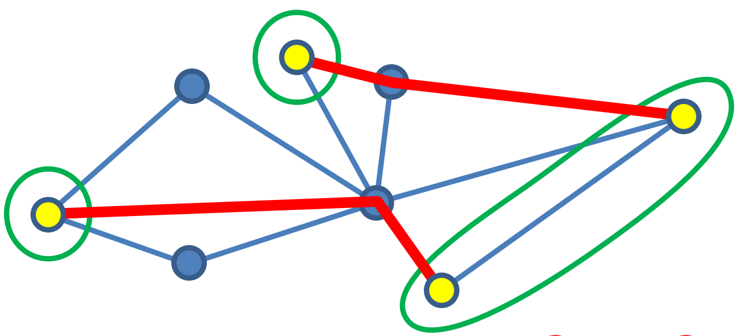
Find **Maximum Number** **of Vertex-Disjoint $\mathcal{S}$-paths**

Shortest Disjoint $\mathcal{S}$-paths Problem

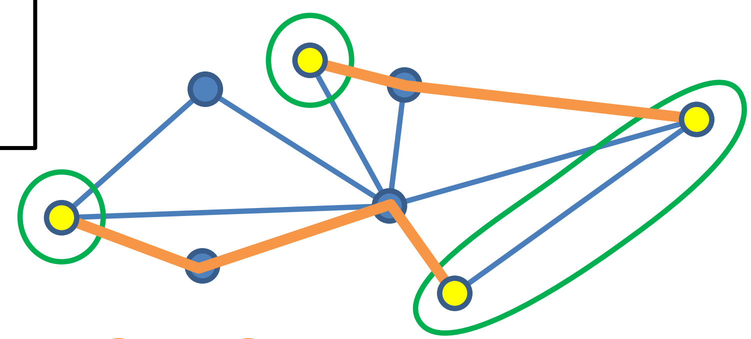
Given $G = (V, E)$: Undirected Graph

$A \subseteq V$: Terminal Set, $\mathcal{S}$: Partition of A

$\ell: E \to \mathbf{R}_{\geq 0}$ Edge Length, $k \in \mathbf{Z}_{>0}$



$$\boxed{\begin{array}{l} \ell \equiv \mathbf{1} \\ k = 2 \end{array}}$$



$$2 + 2 = 4 < 5 = 2 + 3$$

Find **Totally Shortest k Vertex-Disjoint $\mathcal{S}$-paths**

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Linear Matroid Parity Problem

<u>Given</u> $Z \in \mathbf{F}^{r \times 2m}$: Matrix with <u>**Pairing**</u> of Columns

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

<u>Find</u> **Maximum Number of Linearly Independent Column-Pairs**

Linear Matroid Parity Problem

Given $Z \in \mathbf{F}^{r \times 2m}$: Matrix with **Pairing** of Columns

Full Rank
$(\text{rank} = 6)$

$$
\begin{bmatrix}
\boxed{\begin{matrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{matrix}} & \begin{matrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 2 \\ 0 & 1 \\ 1 & 1 \end{matrix}
\end{bmatrix}
$$

Find **Maximum Number of**
Linearly Independent Column-Pairs

Linear Matroid Parity Problem

Given $Z \in \mathbf{F}^{r \times 2m}$: Matrix with **Pairing** of Columns

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NOT Full Rank
$(\mathrm{rank} = 3)$

Find **Maximum Number of Linearly Independent Column-Pairs**

Weighted Linear Matroid Parity Problem

Given $Z \in \mathbf{F}^{r \times 2m}$: Matrix with **Pairing** of Columns

$w\colon [m] \to \mathbf{R}$ **Weight on Column-Pairs**

$$
\begin{array}{c}
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 2 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1
\end{bmatrix} \\
\begin{array}{cccc} \mathbf{2} & \quad \mathbf{3} & \quad 1 & \quad \mathbf{-1} \end{array}
\end{array}
\qquad 4 < 6 \qquad
\begin{array}{c}
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 2 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1
\end{bmatrix} \\
\begin{array}{cccc} \mathbf{2} & \quad \mathbf{3} & \quad \mathbf{1} & \quad -1 \end{array}
\end{array}
$$

Find **Max. Linearly Independent Column-Pairs with Minimum Total Weight**

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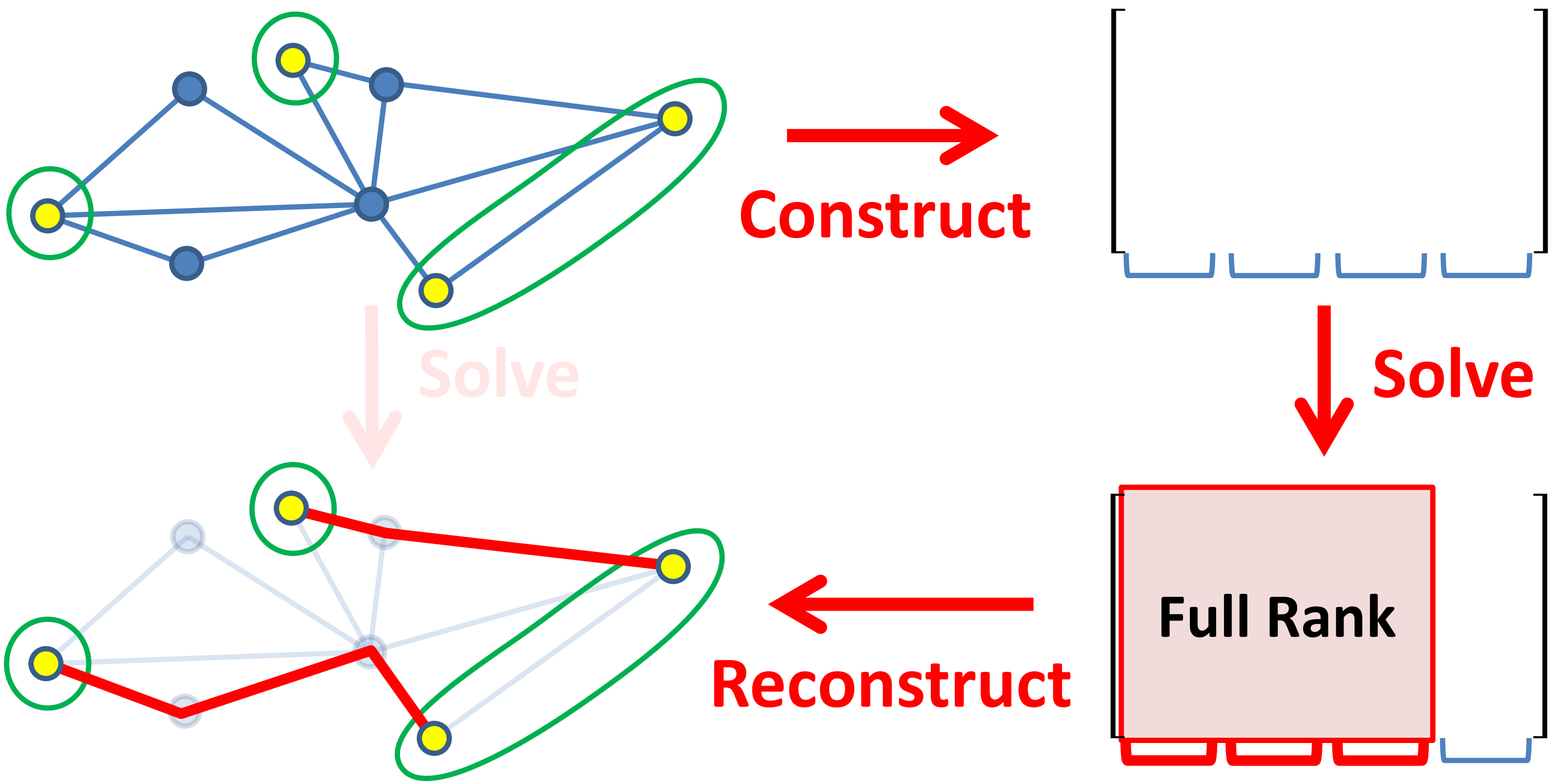
Reduction Sketch

[Lovász 1980][Schrijver 2003]

Disjoint $\mathcal{S}$-paths —Reduce→ Linear Matroid Parity



Reduction Sketch

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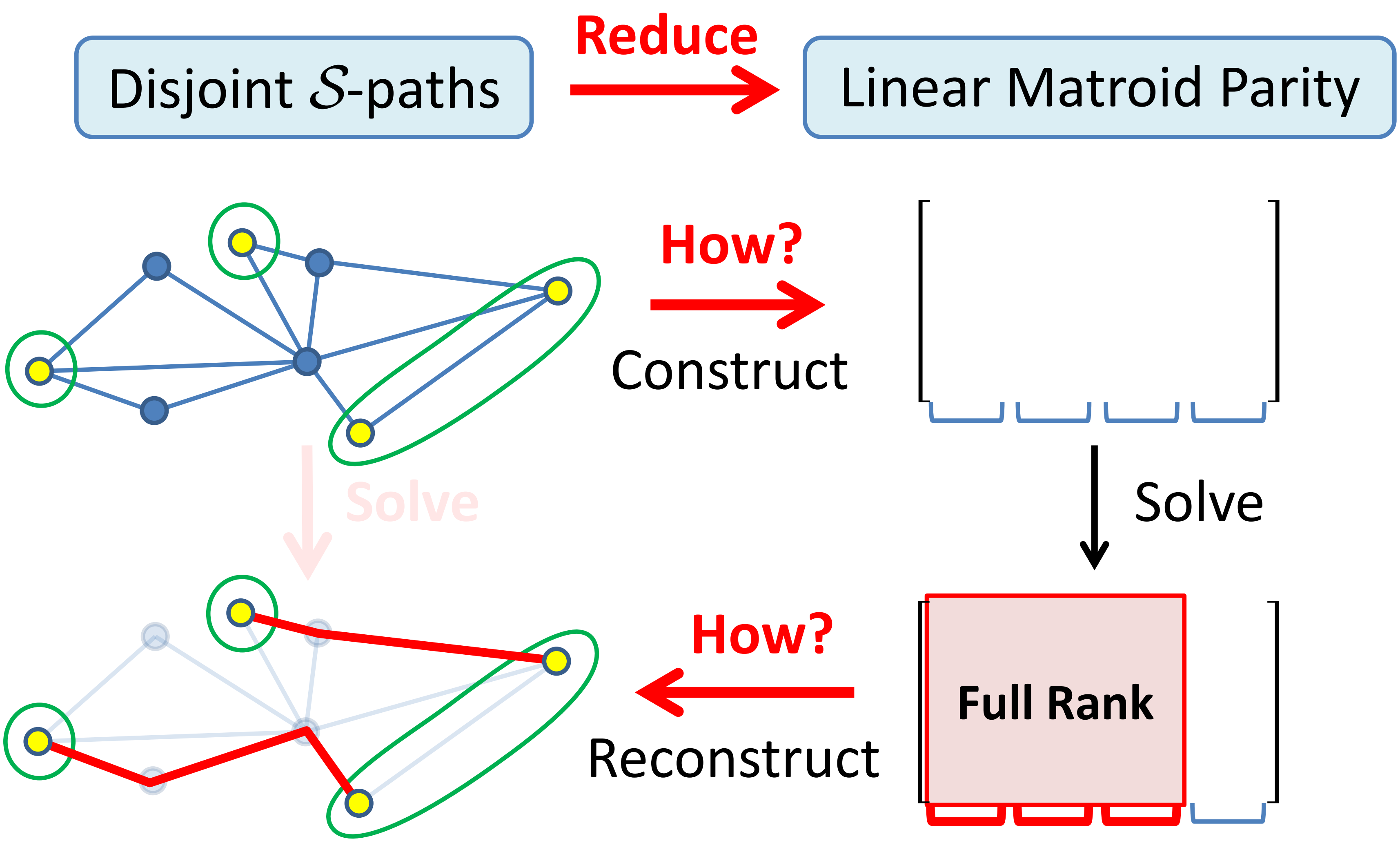
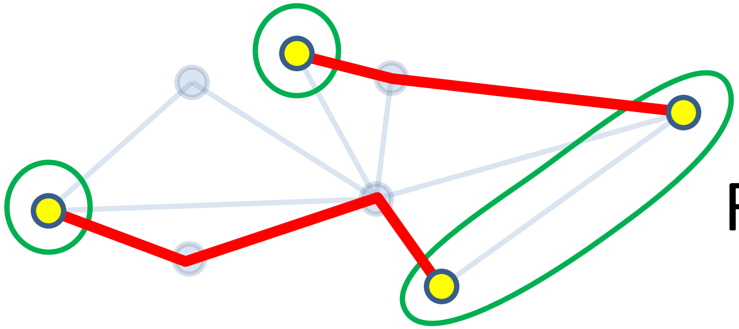
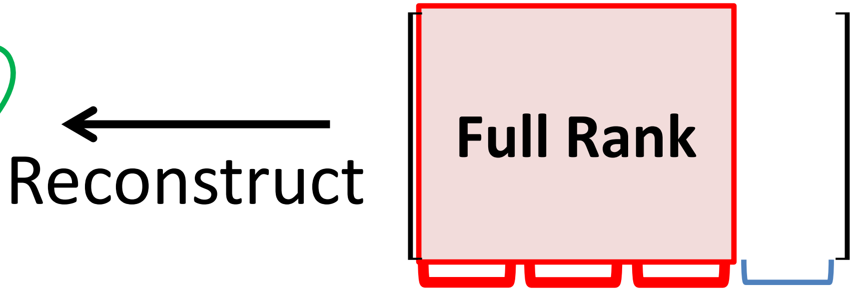


Associated Matrix

[Lovász 1980][Schrijver 2003]

- We want a **Subgraph**

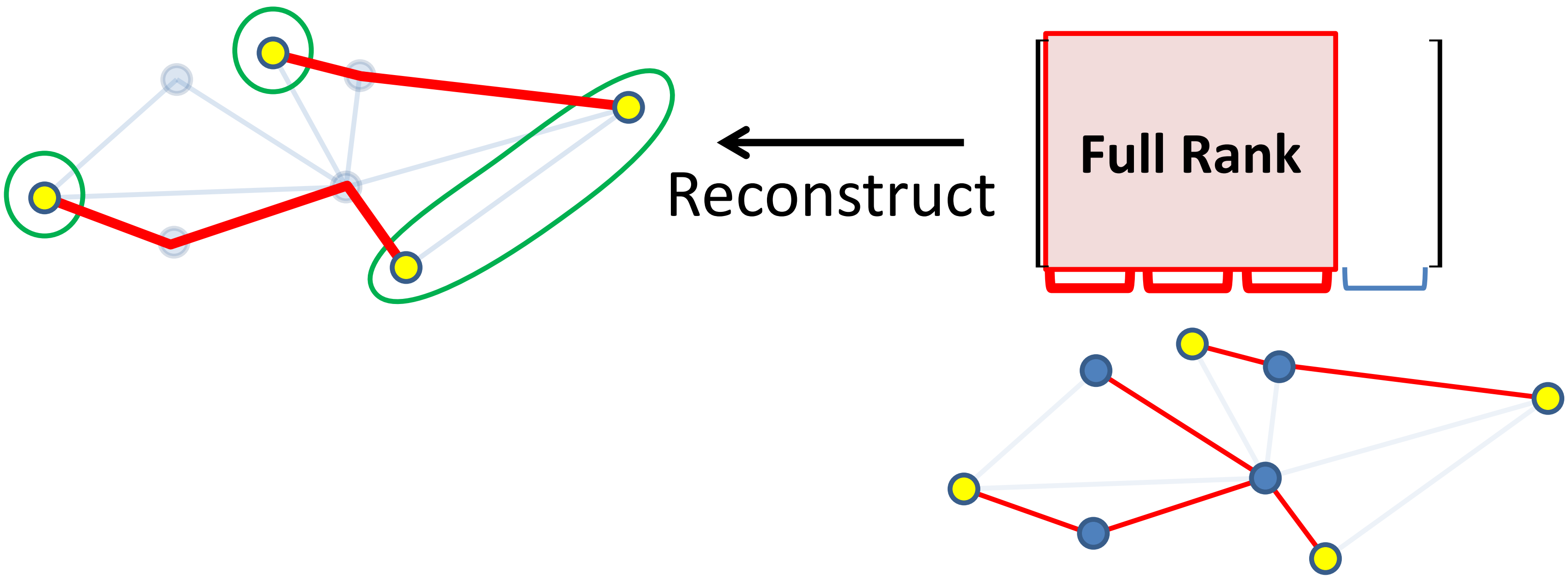




Associated Matrix

[Lovász 1980][Schrijver 2003]

- We want a **Subgraph** → Edge ↔ Column-Pair



Associated Matrix

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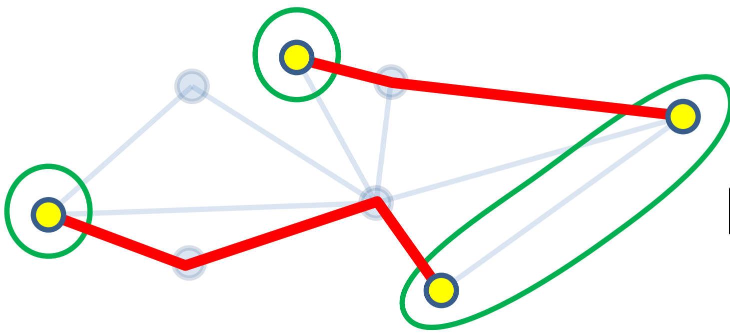
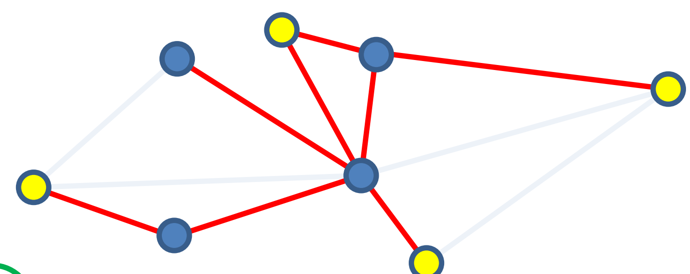
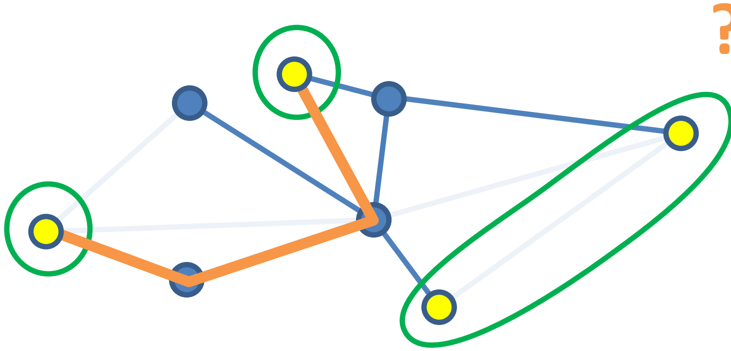


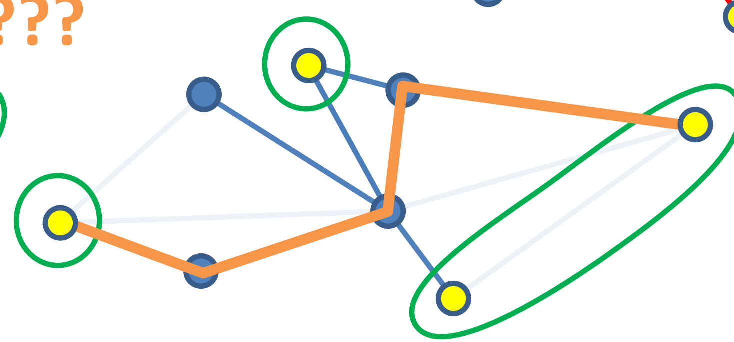
Reconstruct

- We want **Easy** Reconstruction





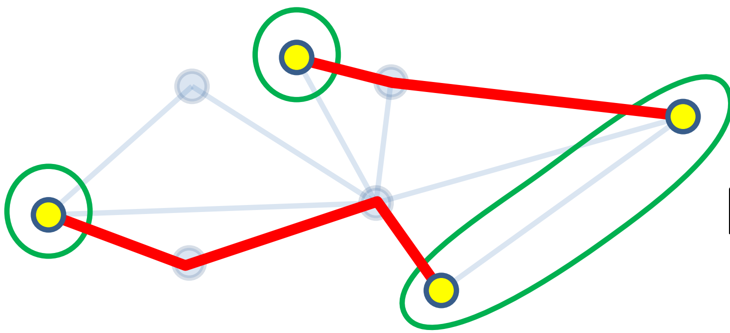
???



Associated Matrix

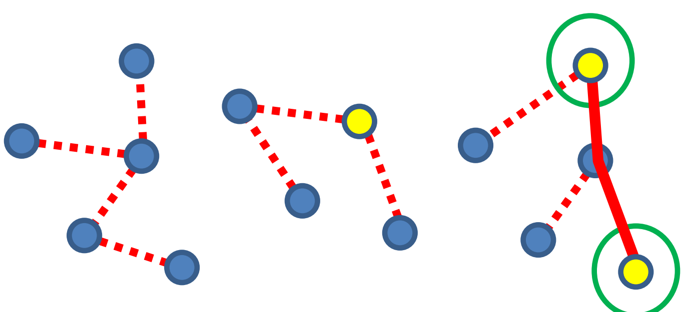
[Lovász 1980][Schrijver 2003]

- We want a **<u>Subgraph</u>** → Edge ↔ Column-Pair



← Reconstruct

- We want <u>**Easy**</u> Reconstruction



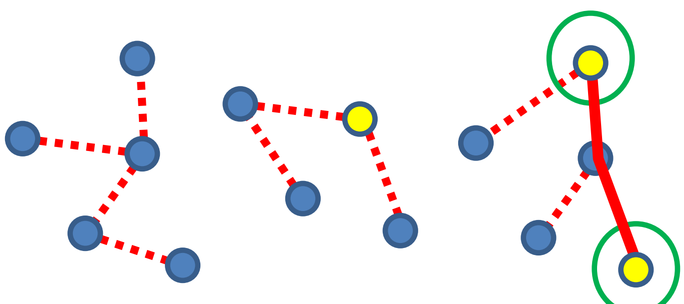
Forest

In each tree **at most one $\mathcal{A}$-path,** which is **$\mathcal{S}$-path** ⟺

Summary of Reduction

[Lovász 1980][Schrijver 2003]

	Disjoint $\mathcal{S}$-paths		Linear Matroid Parity
Cardinality ver.	$\exists$<u>**Max.-Card.**</u> Solution • **Unique** • **Easy to Extract**	$\subseteq$	$\forall$<u>**Max.-Card.**</u> Solution



Forest

In each tree **at most one A-path,** which is **$\mathcal{S}$-path**

$\Longleftrightarrow$

What is Difficult to Extend?

	Disjoint $\mathcal{S}$-paths		Linear Matroid Parity
Cardinality ver.	∃**Max.-Card.** Solution • **Unique** • **Easy to Extract**	⊆	∀**Max.-Card.** Solution
Weighted ver.	**Differ.-Card.** Sols.	⊆	**Fixed-Card.** Sols.
	Extraction does NOT preserve Total Weight		

Nontrivial but Possible to Overcome!

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Ideas to Overcome Difficulties

- Violation of Cardinality Constraint

  Make a Gadget to Convert
  All **Fixed-Card.** Sols. to All **Max.-Card.** Sols.

  cf. $\forall$Max. Sol. in L.M.P. $\supseteq$ $\exists$!Max. $\mathcal{S}$-paths

- Change of Total Weight by Extraction

  Add **Dummy Elements** of **Weight** 0 so that
  They do NOT affect Original $\mathcal{S}$-paths Problem

  $\rightarrow$ Incentive to Use Dummy Elements in W.L.M.P.

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Gadget to Convert Solutions

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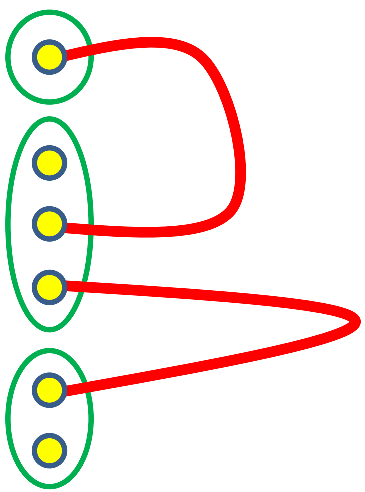
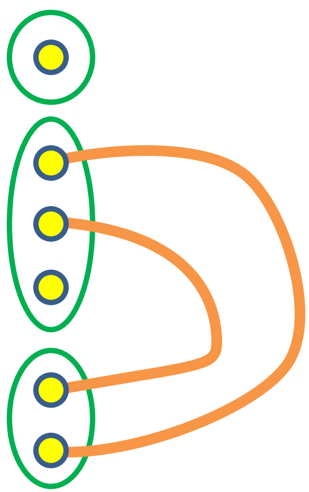
$\boxed{k = 2}$





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Gadget to Convert Solutions

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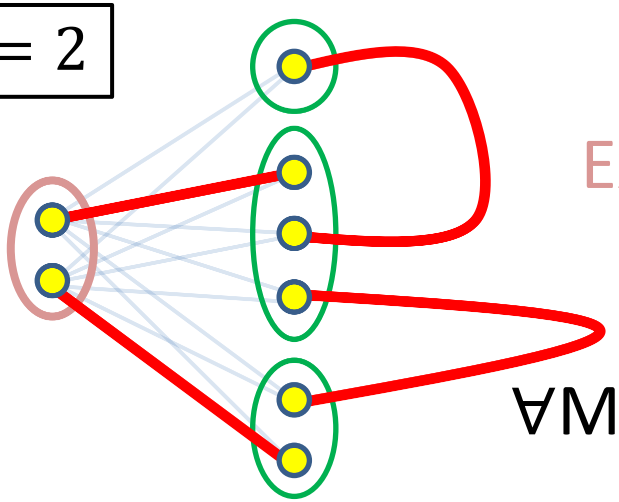
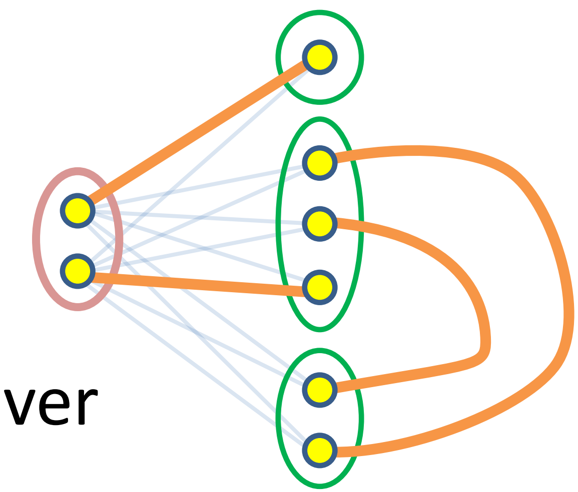


$|A| - 2k$
Extra Terminals

∀Max. $\mathcal{S}$-paths cover
<u>ALL Terminals</u>



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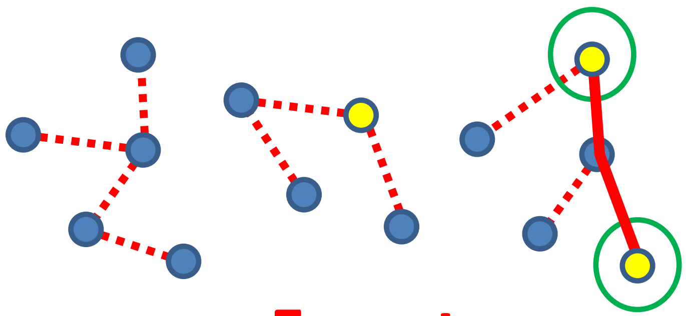
Add Dummy Elements of Weight 0



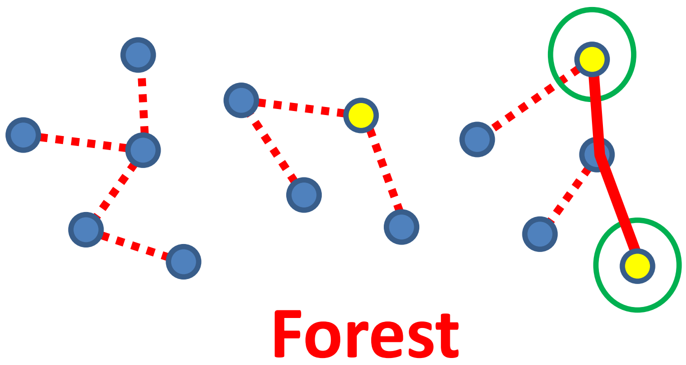
Forest

In each tree **at most one $\mathcal{A}$-path**, which is **$\mathcal{S}$-path** $\Longleftrightarrow$

$\exists!$Max. $\mathcal{S}$-paths $\subseteq$ $\forall$Max. Sol. in L.M.P.

BUT "Total Length < Total Weight" due to **Dotted Edges**

Add Dummy Elements of Weight 0



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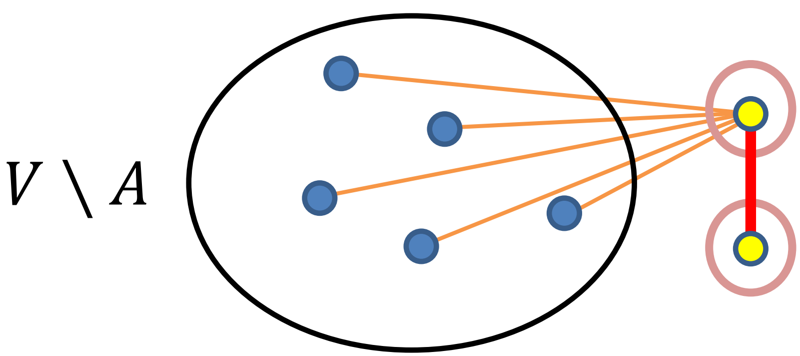


Extra $\mathcal{S}$-path

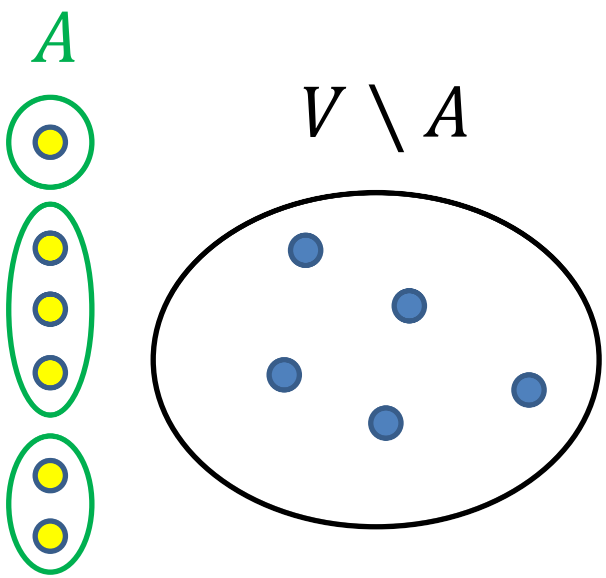
to Rescue

Unused Non-terminals

Summary of Construction

- $|A| - 2k$ Extra Terminals to Rescue Unused Terminals
- An Extra $\mathcal{S}$-path to Rescue Unused Non-terminals

(ALL Extra Edges are of Length 0)



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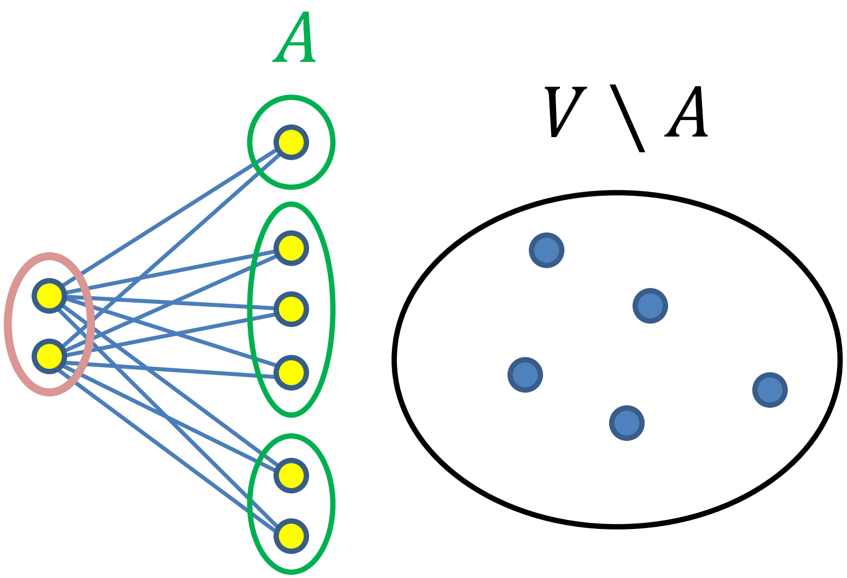

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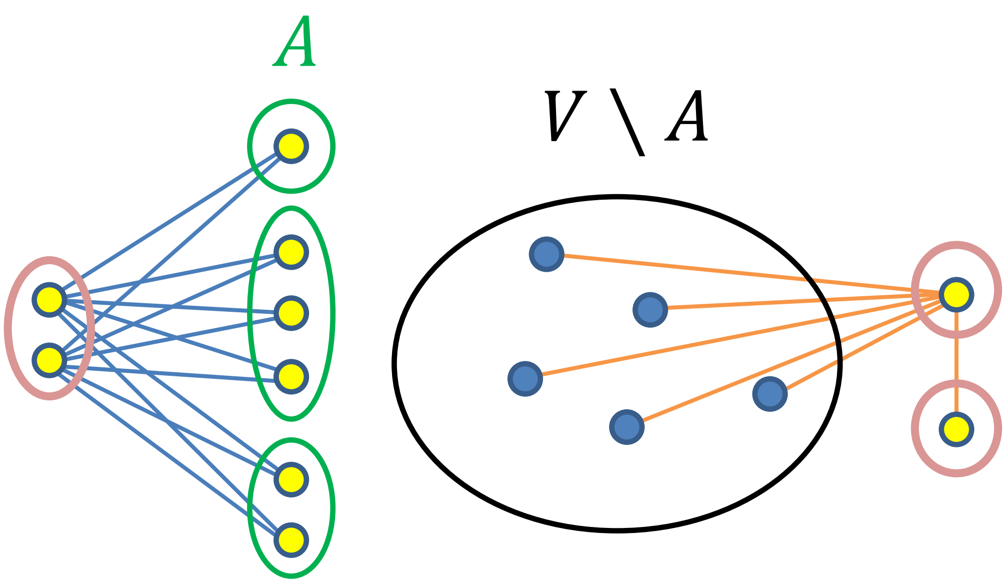

Summary of Construction

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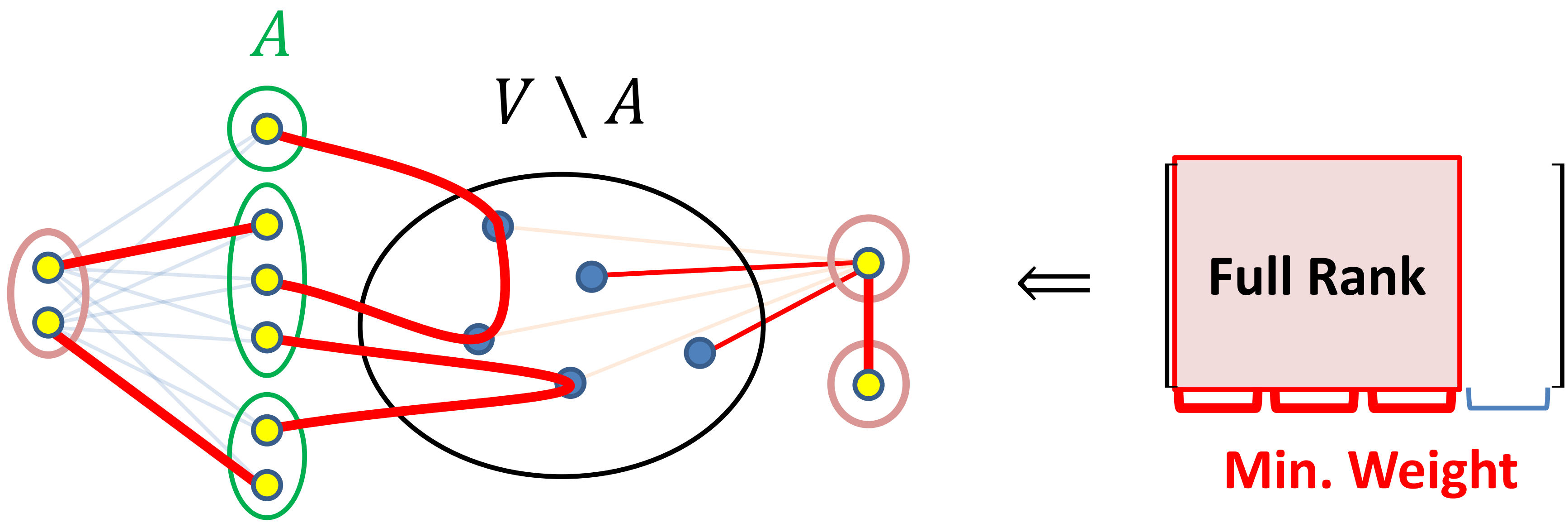
- An Extra $\mathcal{S}$-path to Rescue Unused Non-terminals

(ALL Extra Edges are of Length 0)

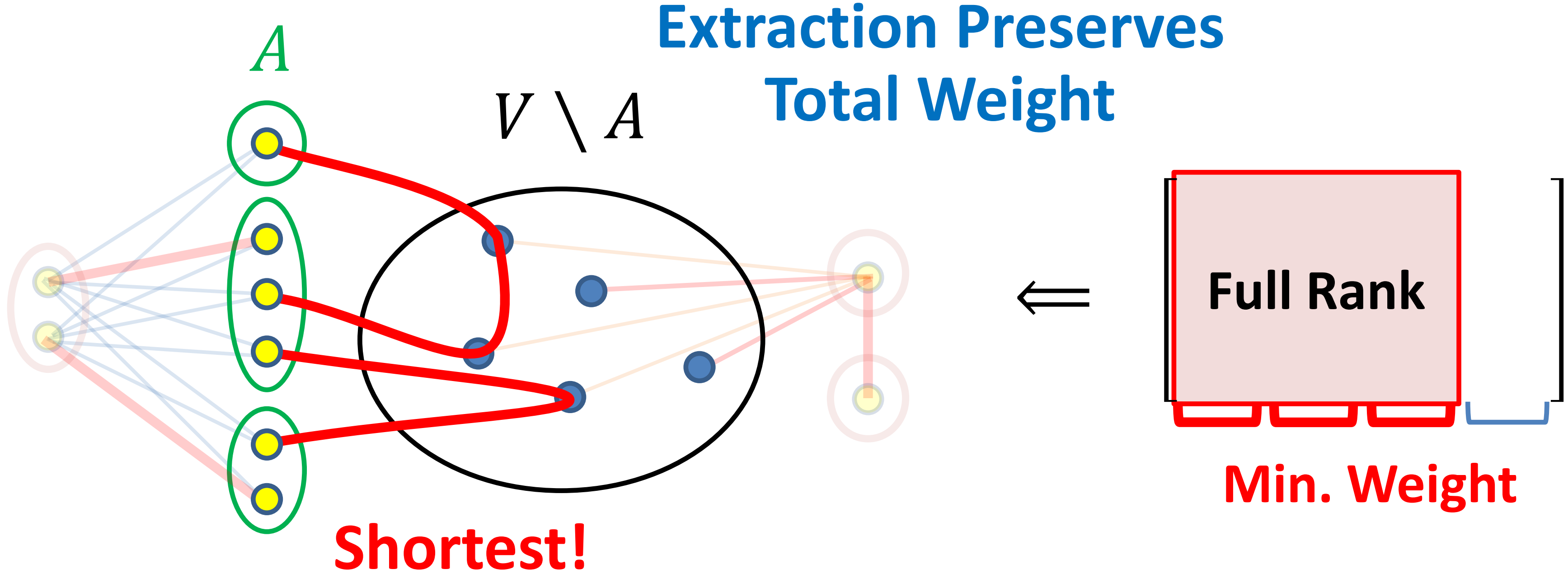


Summary of Construction

- $|A| - 2k$ Extra Terminals to Rescue Unused Terminals
- An Extra $\mathcal{S}$-path to Rescue Unused Non-terminals

(**ALL Extra Edges are of Length 0**)

Extraction Preserves Total Weight



Outline

- Preliminaries
  - Disjoint $\mathcal{S}$-paths
  - Linear Matroid Parity
  - Reduction in Cardinality Case
- Result
  - How to Extend to Weighted Case
- Conclusion

Conclusion

	Disjoint $\mathcal{S}$-paths		Linear Matroid Parity
Cardinality ver.	Polytime [Lovász 1980] [Schrijver 2003]	**Reduce** →	Polytime e.g., [Lovász 1981] [Gabow–Stallmann 1986] [Cheung–Lau–Leung 2016]
Weighted ver.	**Polytime!** [This Talk]	**Reduce!** →	Polytime Announced by [Iwata 2013][Pap 2013]

Conclusion

$$\begin{bmatrix} & r \times 2m & \\ & & \end{bmatrix}$$

	Disjoint $\mathcal{S}$-paths		Linear Matroid Parity
Cardinality ver.	$O(\|V\|^{2.38})$ time [Lovász 1980] [Schrijver 2003]	**Reduce** $\rightarrow$	$O(rm^{1.38})$ time e.g., [Lovász 1981] [Gabow–Stallmann 1986] [Cheung–Lau–Leung 2016]
Weighted ver.	$O(\|V\|^5)$ time [This Talk]	**Reduce!** $\rightarrow$	$O(rm^3)$ time Announced by [Iwata 2013][Pap 2013]

Conclusion

- Shortest Disjoint $\mathcal{S}$-paths Problem
  is solved in $\mathrm{O}(|V|^5)$ time via Weighted L.M.P.

- This result can be extended to
  Packing Non-zero A-paths in Group-Labeled Graphs
  under some Group Representability Condition [Y. 2016]

Q. More Efficient or Direct Algorithms?

Q. "Non-zero & Weighted" is Generally in P?