# On Applications
# of Weighted Linear Matroid Parity

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# Applications of Linear Matroid Parity

- Maximum Forests in $3$-Uniform Hypergraphs [Lovász 1980]
- Maximum Disjoint $\mathcal{S}$-paths [Lovász 1980][Schrijver 2003]
- Minimum Pinning-Down Points to Make Planar Structures Rigid [Lovász 1980]
- Minimum Feedback Vertex Sets in (Sub)Cubic Graphs [Ueno–Kajitani–Gotoh 1988]
- Maximum-Genus Embedding of Graphs [Furst–Gross–McGeoch 1988]

etc.

# Analogy in Weighted Situations?

| | Some Problem | Linear Matroid Parity |
|---|---|---|
| Unweighted (Cardinality) ver. | Polytime ← **via Reduction** | Polytime<br>[Lovász 1978] |
| Weighted ver. | Polytime?? ⇠ **???** | **Polytime**<br>[Iwata–Kobayashi 2017]<br>(Also Announced by [Pap 2013]) |

# Outline

- Preliminaries
- Disjoint $\mathcal{S}$-paths
  - Background
  - Extension Trick
- Feedback Vertex Sets in (Sub)Cubic Graphs
  - Background
  - Extension Trick
- Conclusion

# Outline

- Preliminaries  (What is Difficult?)
- Disjoint $\mathcal{S}$-paths
  - Background
  - Extension Trick
- Feedback Vertex Sets in (Sub)Cubic Graphs
  - Background
  - Extension Trick
- Conclusion

# Linear Matroid Parity Problem

**Given** $Z \in \mathbf{F}^{r \times 2m}$: Matrix with **Lines** (Pairing of Columns)

**Find** **Maximum Number of Linearly Independent Lines**

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

**Column Full Rank**

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

# Weighted Linear Matroid Parity Problem

**Given** $Z \in \mathbf{F}^{r \times 2m}$: Matrix with **Lines** (Pairing of Columns)

$w : [m] \to \mathbf{R}$ **Weight on Lines**

**Find** **Parity Base** **of Minimum Weight**

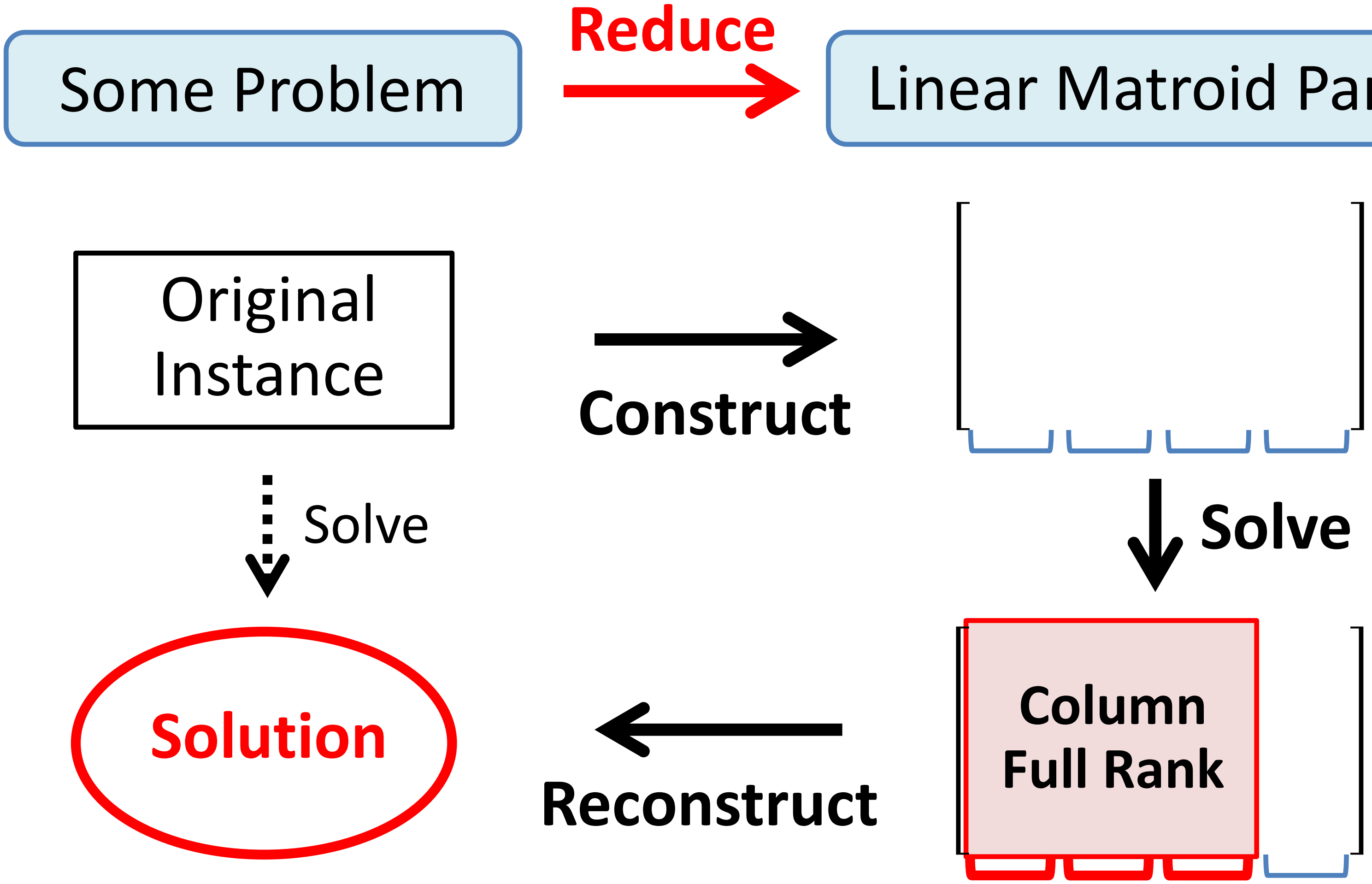
Line Subset consistng of a **Basis**

$$\begin{array}{c} \left[\begin{array}{cccccccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{array}\right] \\ \begin{array}{cccc} \mathbf{2} & \mathbf{3} & 1 & \mathbf{-1} \end{array} \end{array} \quad \Updownarrow \quad \begin{array}{c} \left[\begin{array}{cccccccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{array}\right] \\ \begin{array}{cccc} \mathbf{2} & \mathbf{3} & \mathbf{1} & -1 \end{array} \end{array}$$

**Non-singular**

$4 < 6$

# Reduction Sketch (Unweighted Case)



Some Problem

**Reduce**

Linear Matroid Parity

Original Instance

**Construct**

Solve

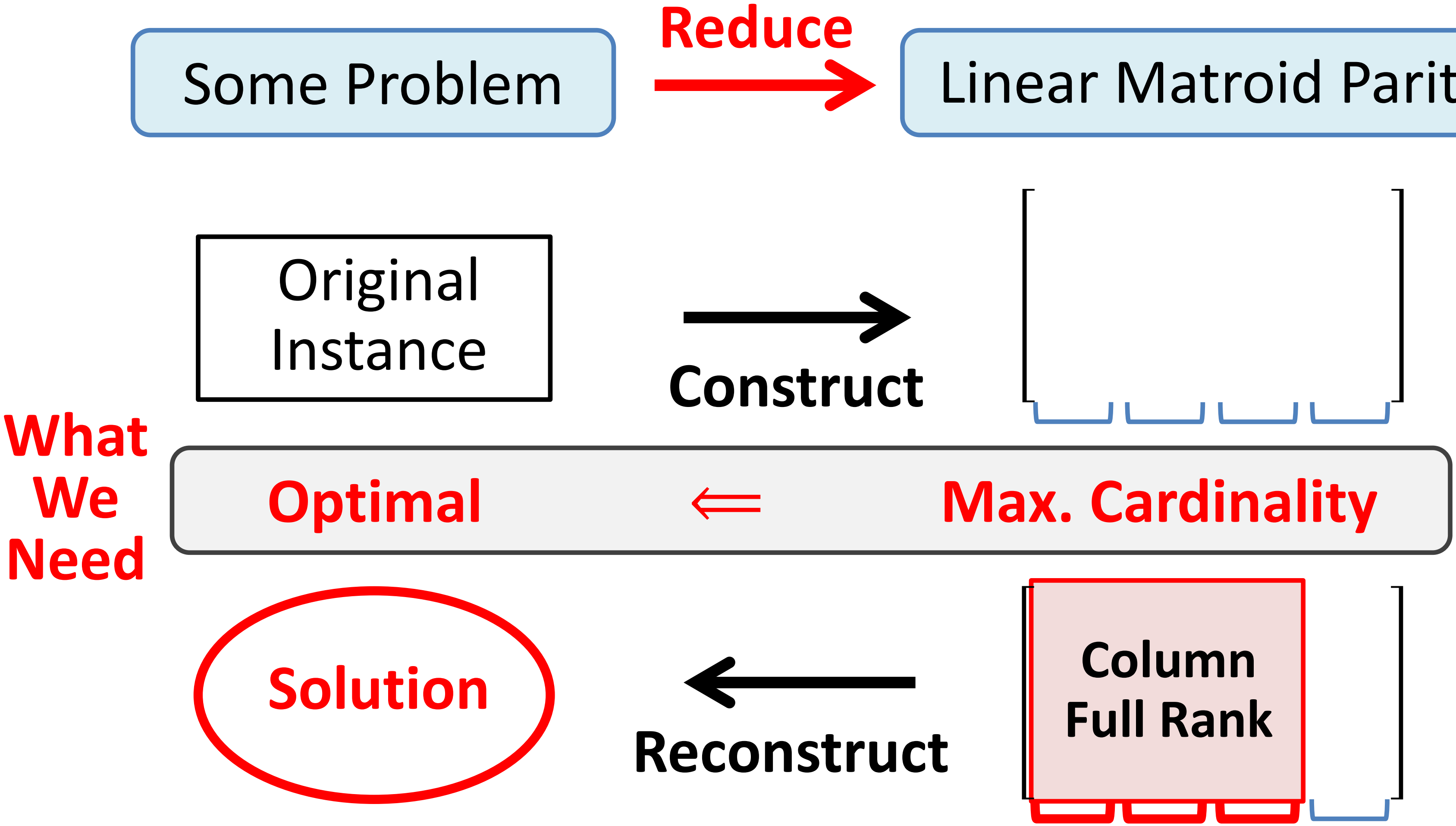
**Solve**

**Solution**

**Reconstruct**

**Column Full Rank**

# Reduction Sketch (Unweighted Case)



Some Problem

**Reduce** →

Linear Matroid Parity

Original Instance

**Construct**

**What We Need**

**Optimal** $\Leftarrow$ **Max. Cardinality**

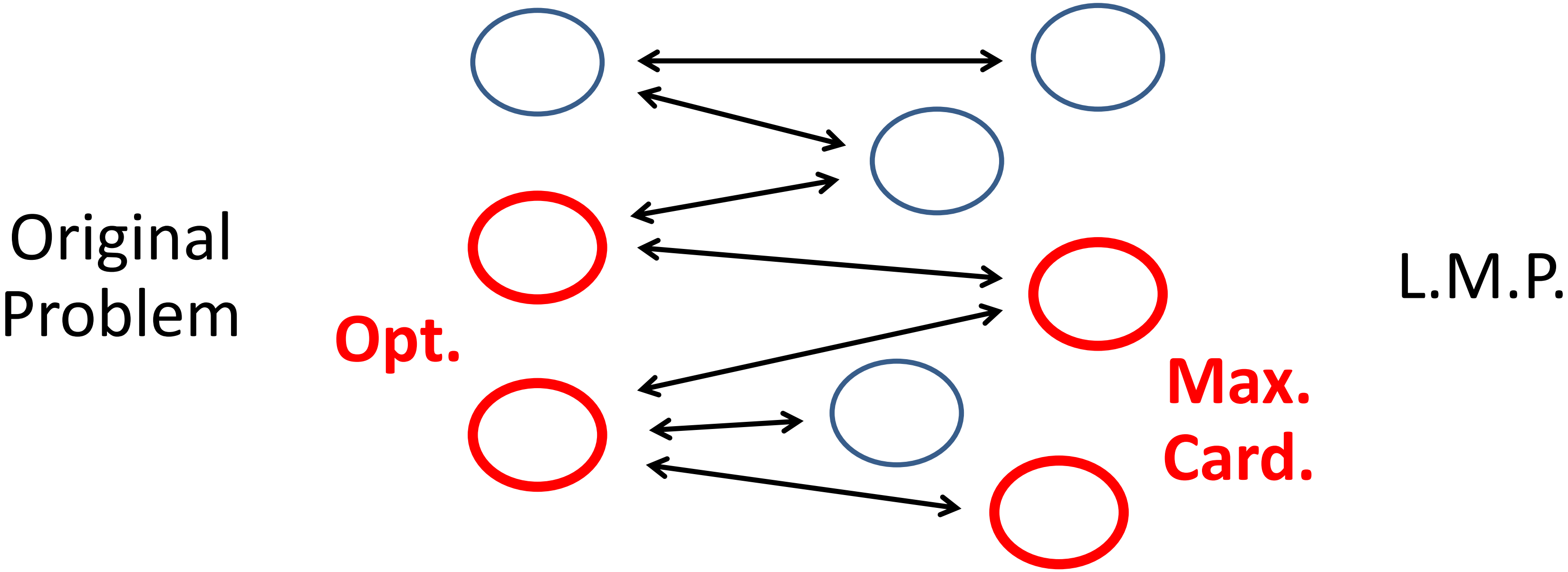
**Solution**

**Reconstruct**

**Column Full Rank**

# General Difficulty

- Solution Correspondence may NOT be One-to-One
- Weights of Solutions may NOT be Preserved



# General Difficulty

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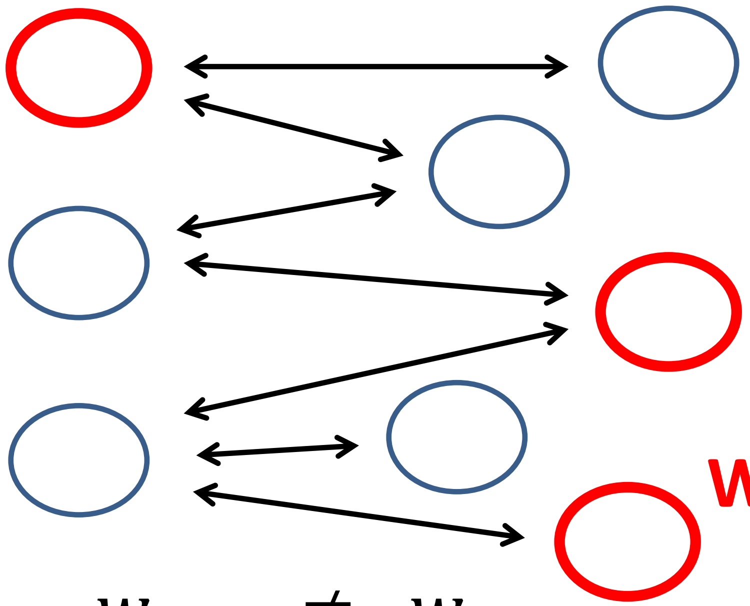


**Opt. Weight**

Original Problem

L.M.P.

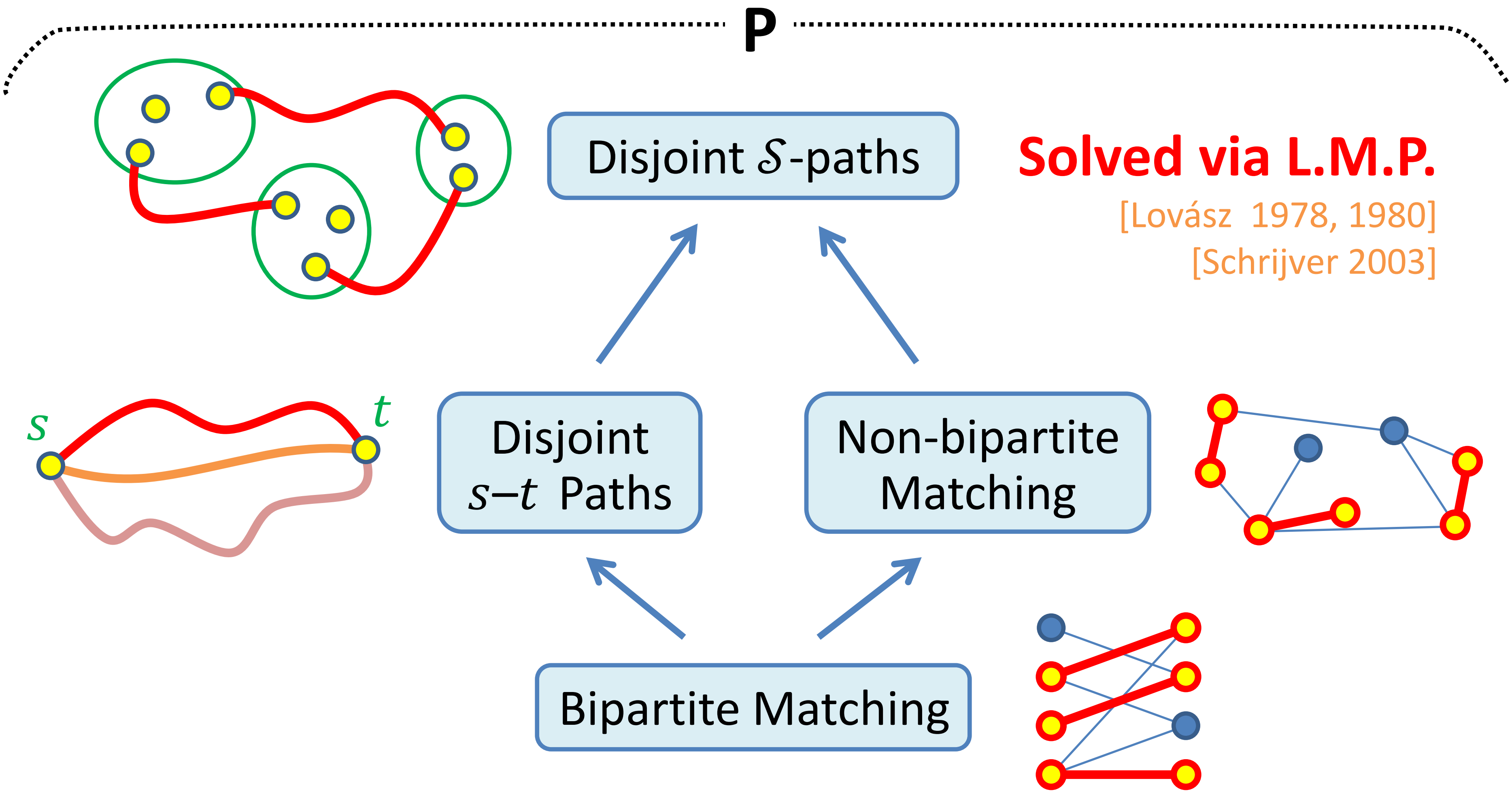
**Opt. Weight**

$$w_{\mathrm{Org}} \neq w_{\mathrm{LMP}}$$

# Outline

- Preliminaries
- Disjoint $\mathcal{S}$-paths
  - Background
  - Extension Trick
- Feedback Vertex Sets in (Sub)Cubic Graphs
  - Background
  - Extension Trick
- Conclusion

# Overview on Cardinality Maximization



Disjoint $\mathcal{S}$-paths

**Solved via L.M.P.**

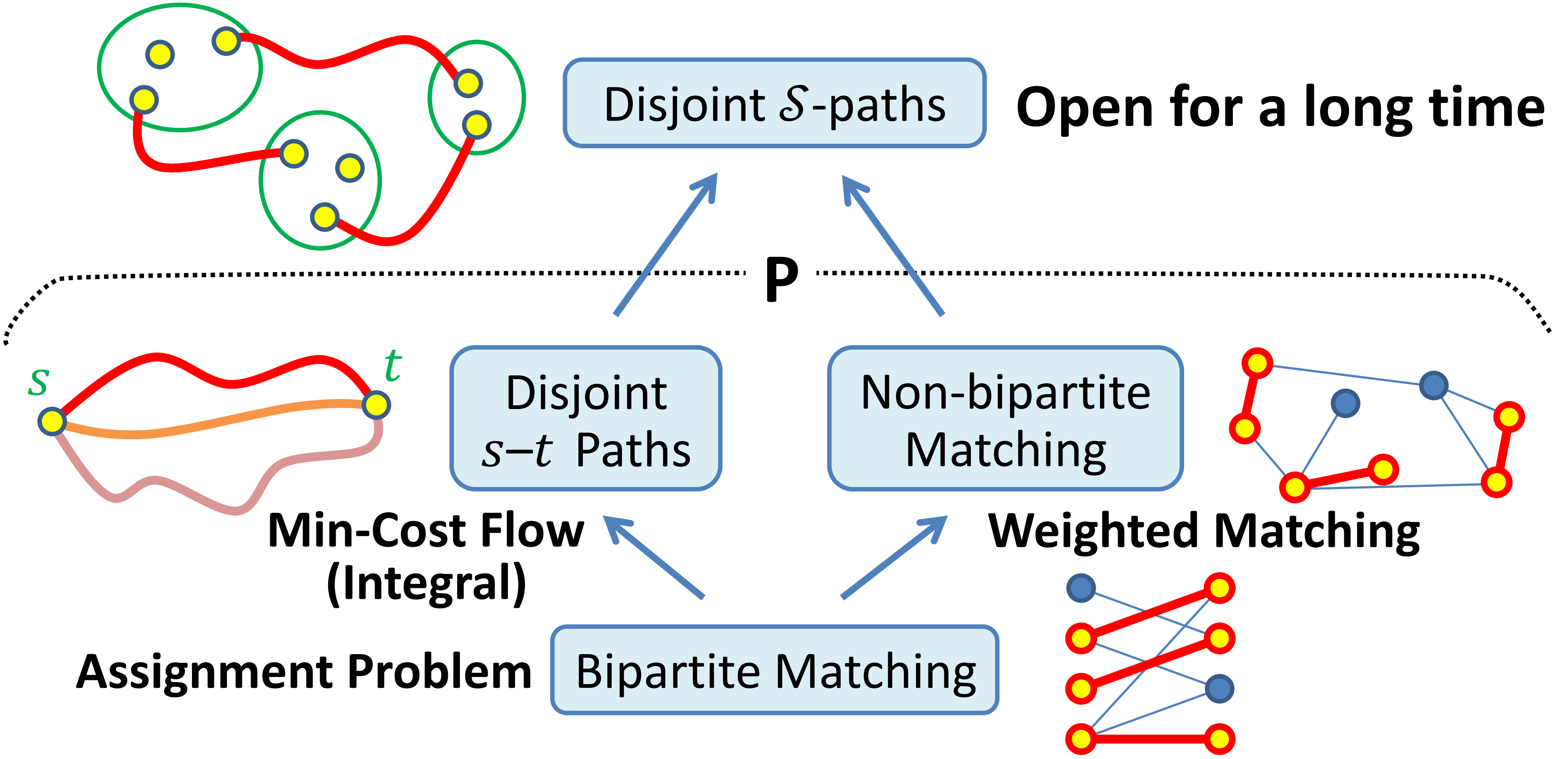
[Lovász 1978, 1980]
[Schrijver 2003]

Disjoint $s$–$t$ Paths

Non-bipartite Matching

Bipartite Matching

# Overview on Cost Minimization



Disjoint $\mathcal{S}$-paths

**Open for a long time**

**P**

Disjoint $s$–$t$ Paths

Non-bipartite Matching

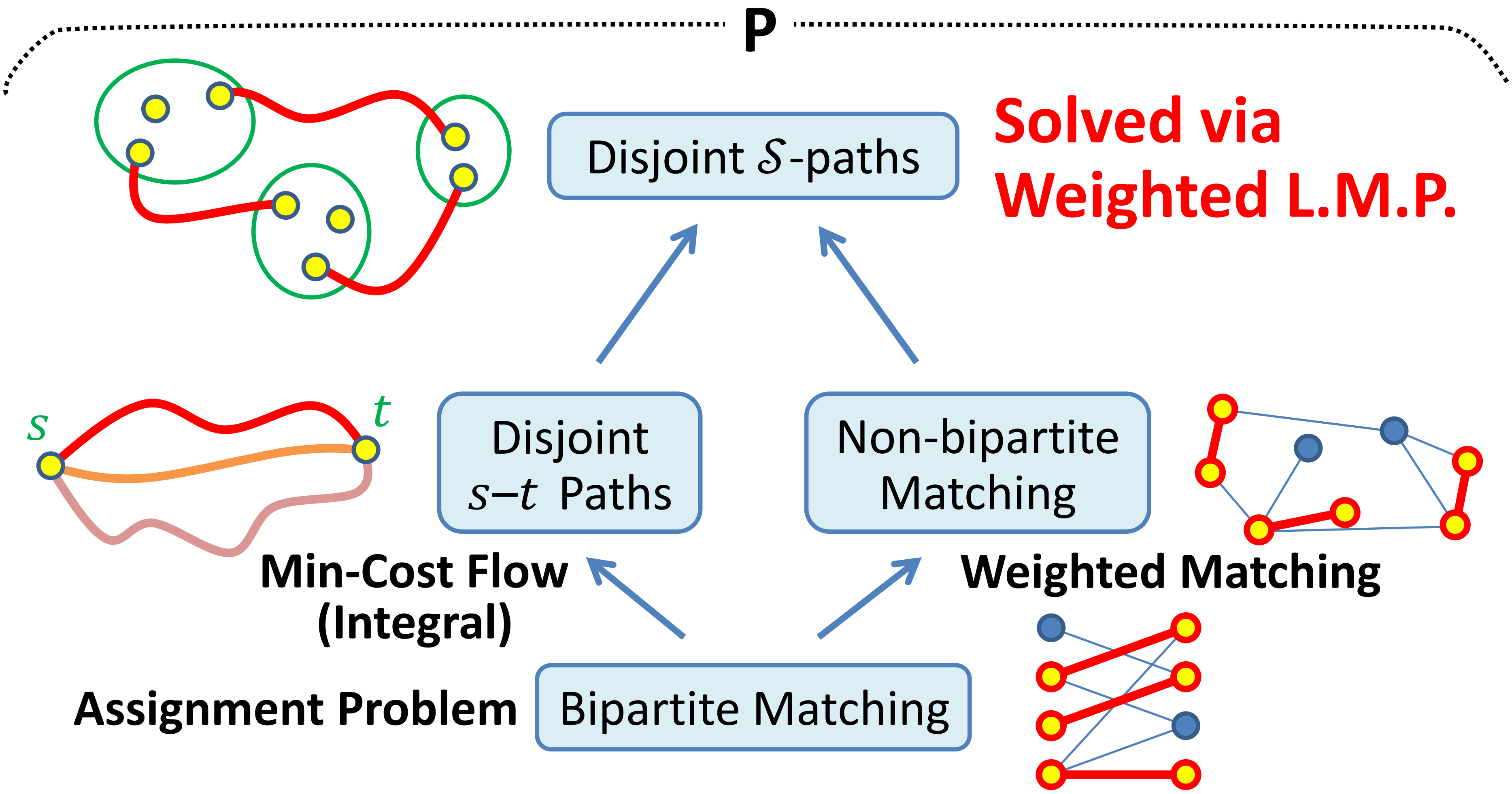
**Min-Cost Flow (Integral)**

**Weighted Matching**

**Assignment Problem**

Bipartite Matching

# Overview on Cost Minimization



**P**

Disjoint $\mathcal{S}$-paths

**Solved via Weighted L.M.P.**

Disjoint $s$–$t$ Paths

**Min-Cost Flow (Integral)**

Non-bipartite Matching

**Weighted Matching**

**Assignment Problem**

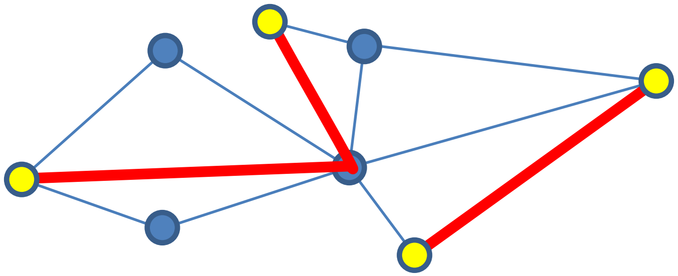
Bipartite Matching

# $A$-paths and $\mathcal{S}$-paths

$G = (V, E)$: Undirected Graph

$A \subseteq V$: Terminal Set

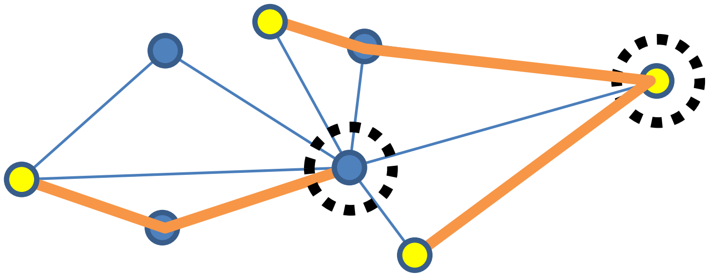
$A$-paths



● $\in A$

● $\notin A$

NOT $A$-paths



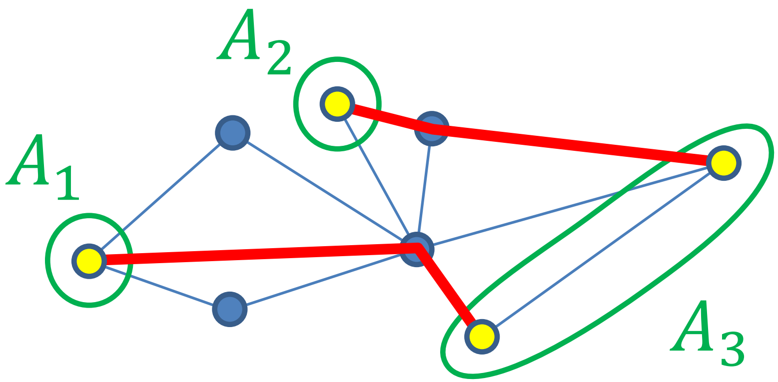
# $A$-paths and $\mathcal{S}$-paths

$G = (V, E)$: Undirected Graph

$A \subseteq V$: Terminal Set

$\mathcal{S} = \{A_1, A_2, \ldots, A_k\}$: **Partition** of $A$

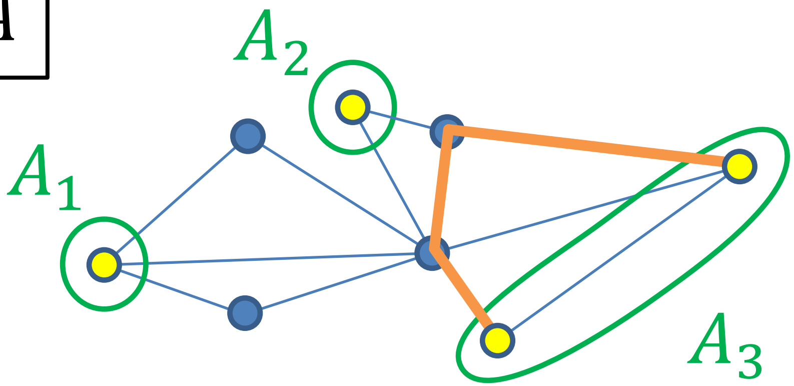
$\mathcal{S}$-paths



● $\in A$

● $\notin A$

NOT $\mathcal{S}$-path



# Disjoint $\mathcal{S}$-paths Problem

**Given** $G = (V, E)$: Undirected Graph

$A \subseteq V$: Terminal Set, $\mathcal{S}$: Partition of $A$

**Find** **Maximum Number** **of Vertex-Disjoint $\mathcal{S}$-paths**

including Terminals

# Shortest Disjoint $\mathcal{S}$-paths Problem

**Given** $G = (V, E)$: Undirected Graph

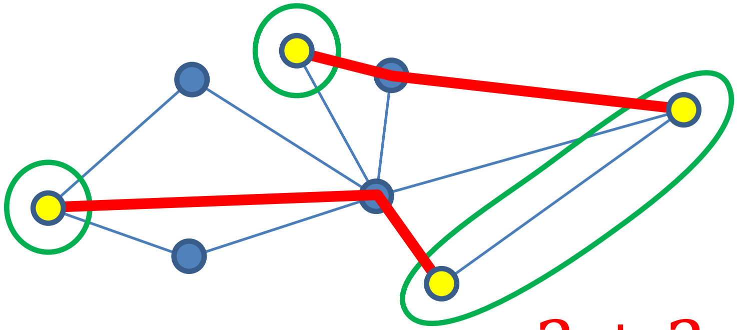
$A \subseteq V$: Terminal Set, $\mathcal{S}$: Partition of $A$

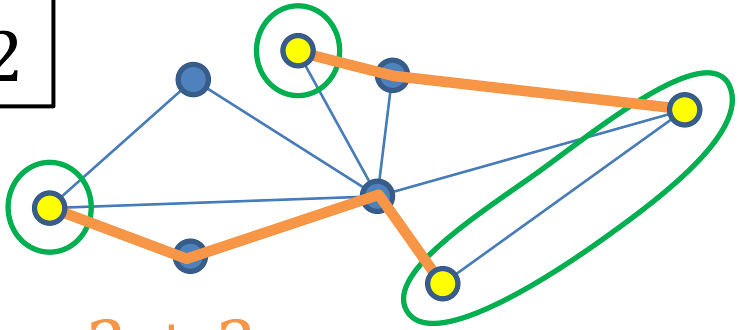
$\ell : E \to \mathbf{R}_{\geq 0}$ **Edge Length**, $k \in \mathbf{Z}_{>0}$

**Find** **Totally Shortest** $k$ **Vertex-Disjoint** $\mathcal{S}$**-paths**

**Ex.**

$$\boxed{\begin{array}{l} \ell \equiv \mathbf{1} \\ k = 2 \end{array}}$$

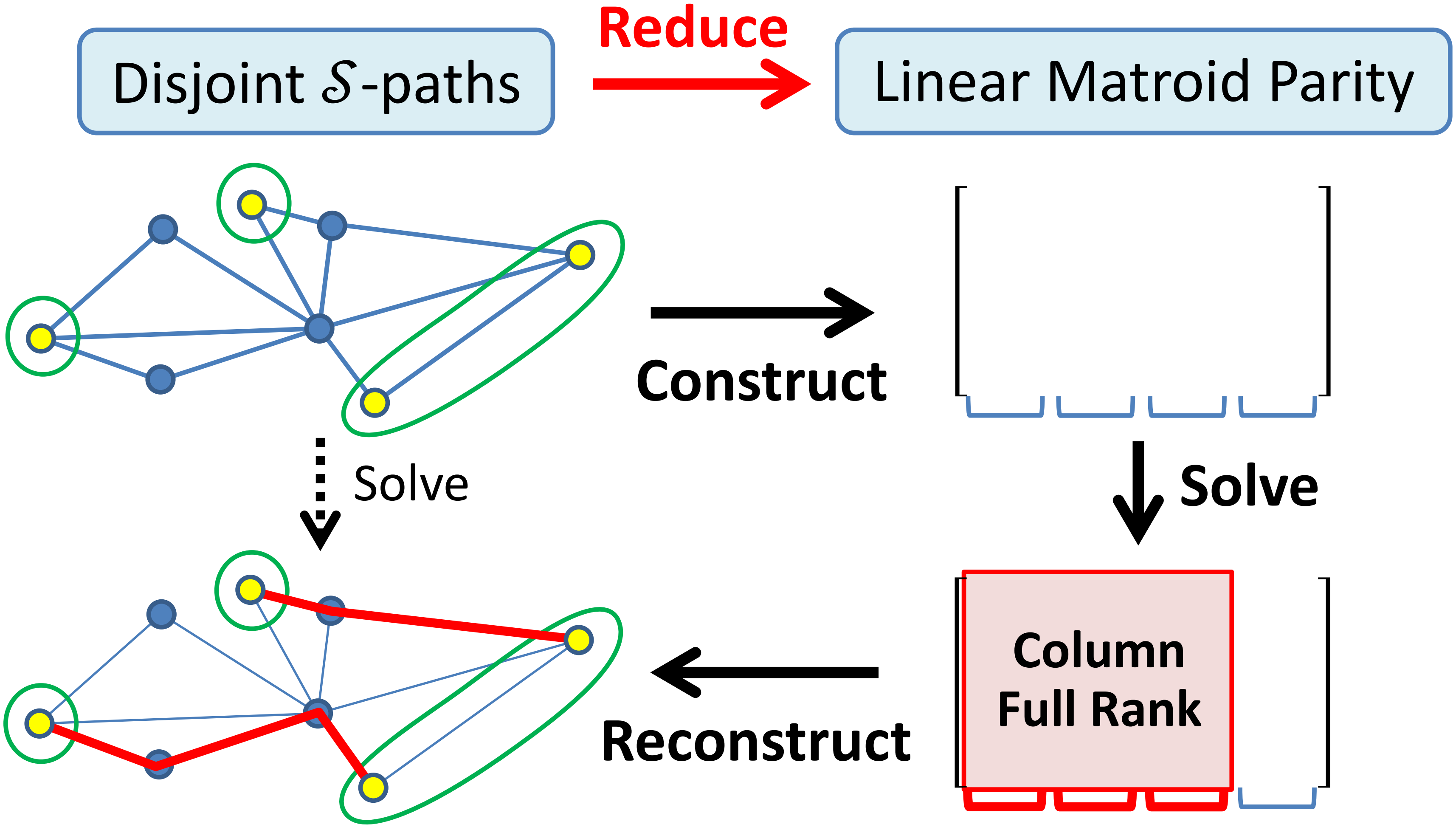




$$2 + 2 = 4 < 5 = 2 + 3$$

# Disjoint $\mathcal{S}$-paths → Linear Matroid Parity

[Lovász 1980][Schrijver 2003]



Disjoint $\mathcal{S}$-paths

**Reduce**

Linear Matroid Parity

**Construct**

Solve

**Solve**

**Column Full Rank**

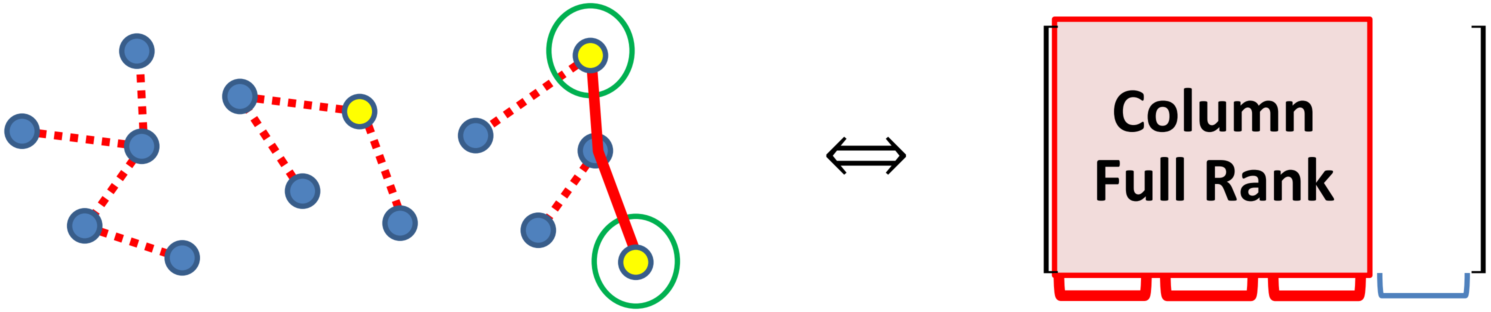
**Reconstruct**

# Disjoint $\mathcal{S}$-paths → Linear Matroid Parity

[Lovász 1980][Schrijver 2003]

**Thm.** $\forall (G=(V,E),A,\mathcal{S})$, ∃ L.M.P. Instance s.t.

- The **Line** set is the **Edge** set $E$
- $F \subseteq E$ is **Feasible** if and only if
  - the **Subgraph** $G[F]$ is a **Forest**, and
  - each tree has **at most one $A$-path**, which is an **$\mathcal{S}$-path**



$\Longleftrightarrow$

**Column Full Rank**

# Disjoint $\mathcal{S}$-paths → Linear Matroid Parity

[Lovász 1980][Schrijver 2003]

**Thm.** $\forall (G = (V, E), A, \mathcal{S})$, $\exists$ L.M.P. Instance s.t.

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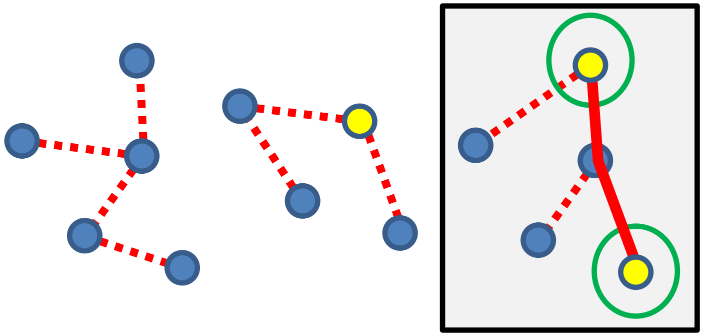
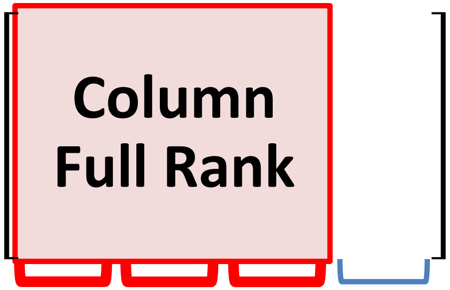
**Max. #(This Type)** ⟸ **Max. Card.**



⟺



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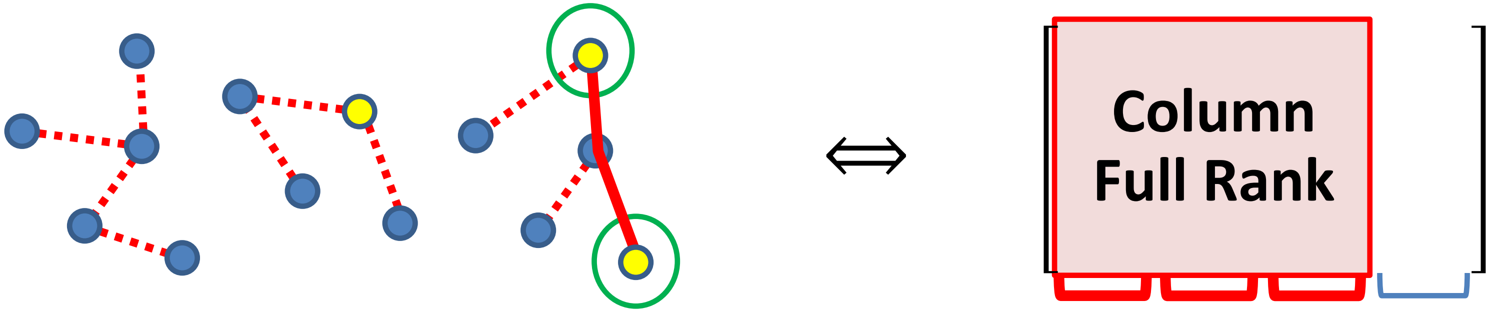
Total Length of $\mathcal{S}$-paths & **Dotted Edges** $=$ **Weight**



# Outline

- Preliminaries
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  - Background
  - Extension Trick (Constructing Auxiliary Instance)
- Feedback Vertex Sets in (Sub)Cubic Graphs
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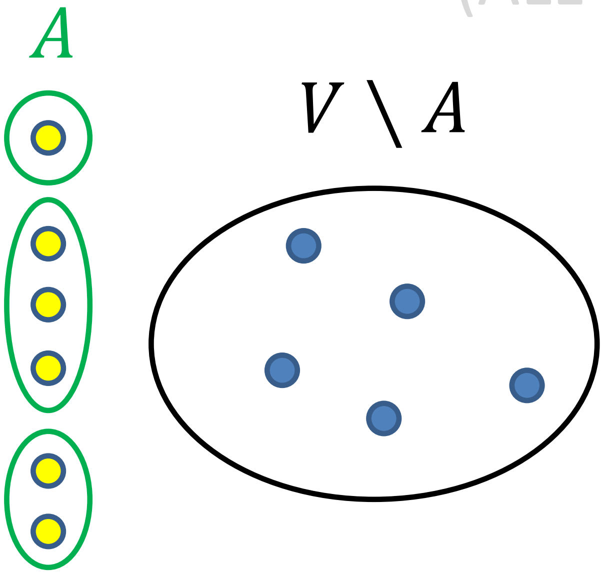
# Construction of Auxiliary Graph

- $|A| - 2k$ Extra Terminals to Rescue Unused Terminals (Because we want to Find $k$ Vertex-Disjoint $\mathcal{S}$-paths)
- An Extra $\mathcal{S}$-path to Rescue Unused Non-terminals

(ALL Extra Edges are of Length 0)



# Construction of Auxiliary Graph

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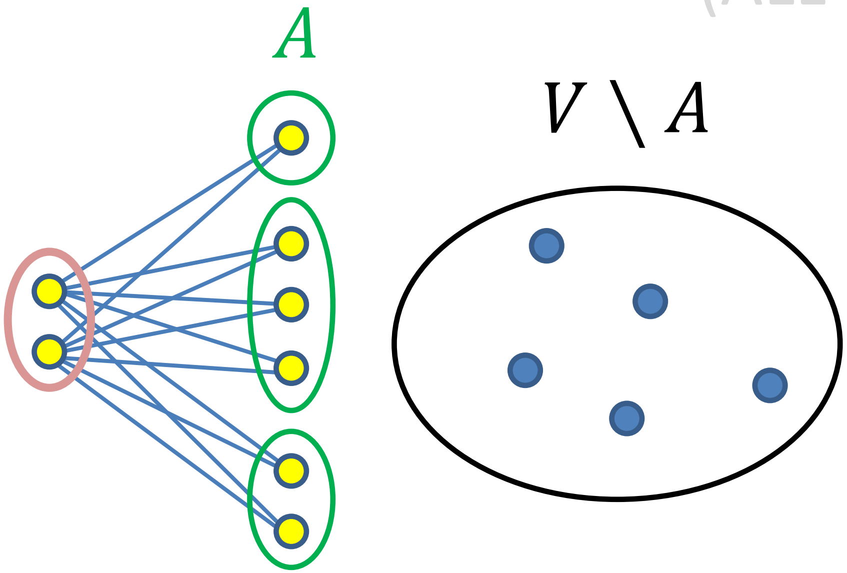

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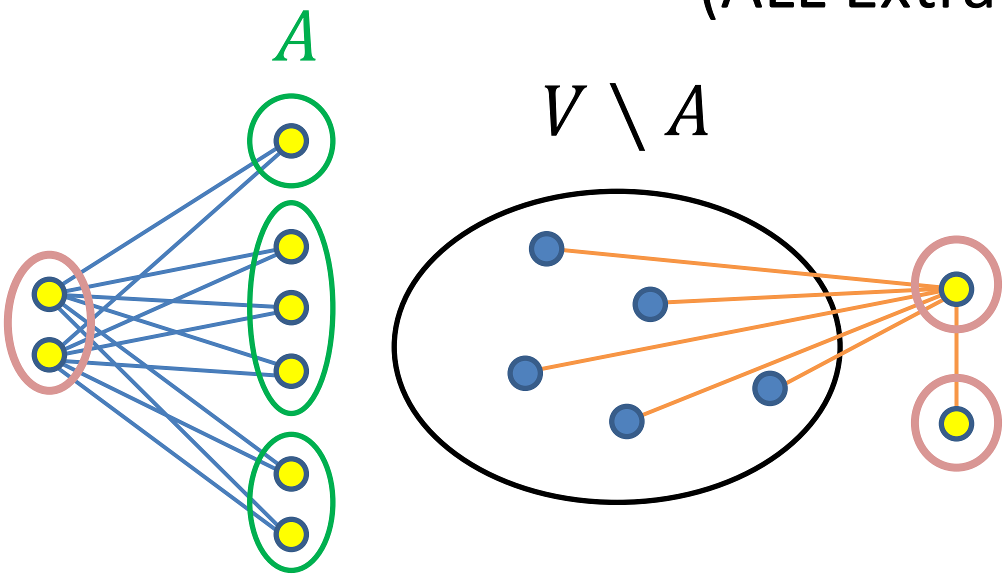

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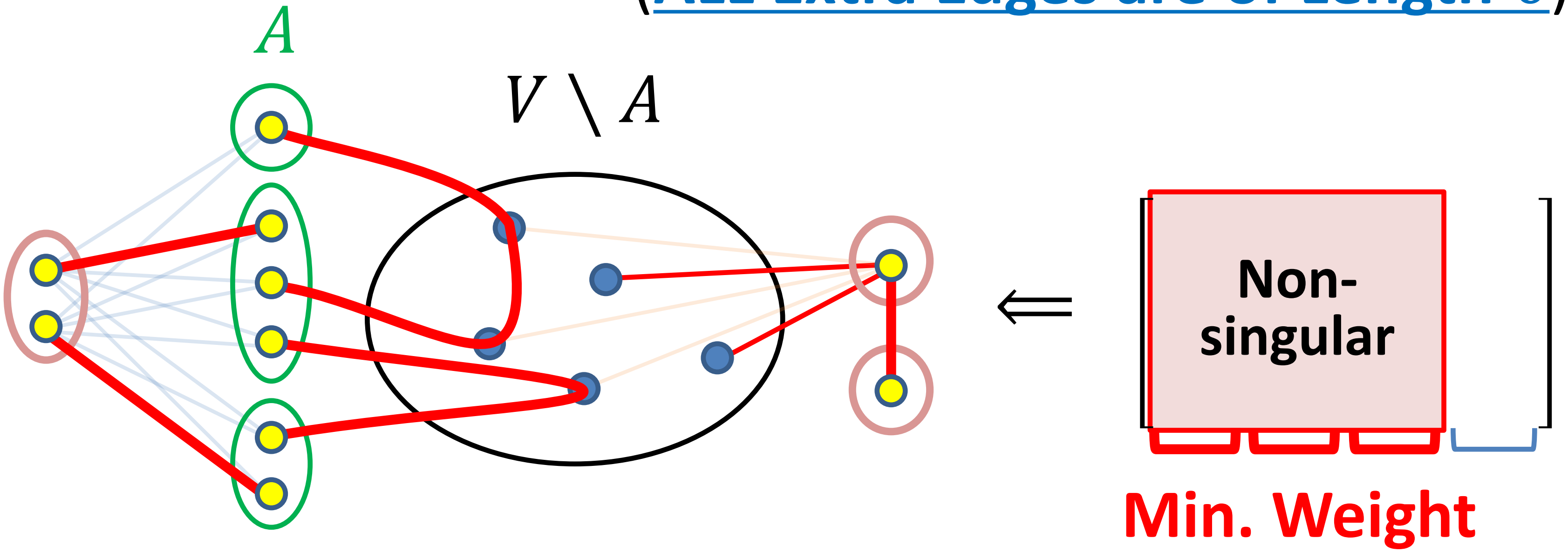

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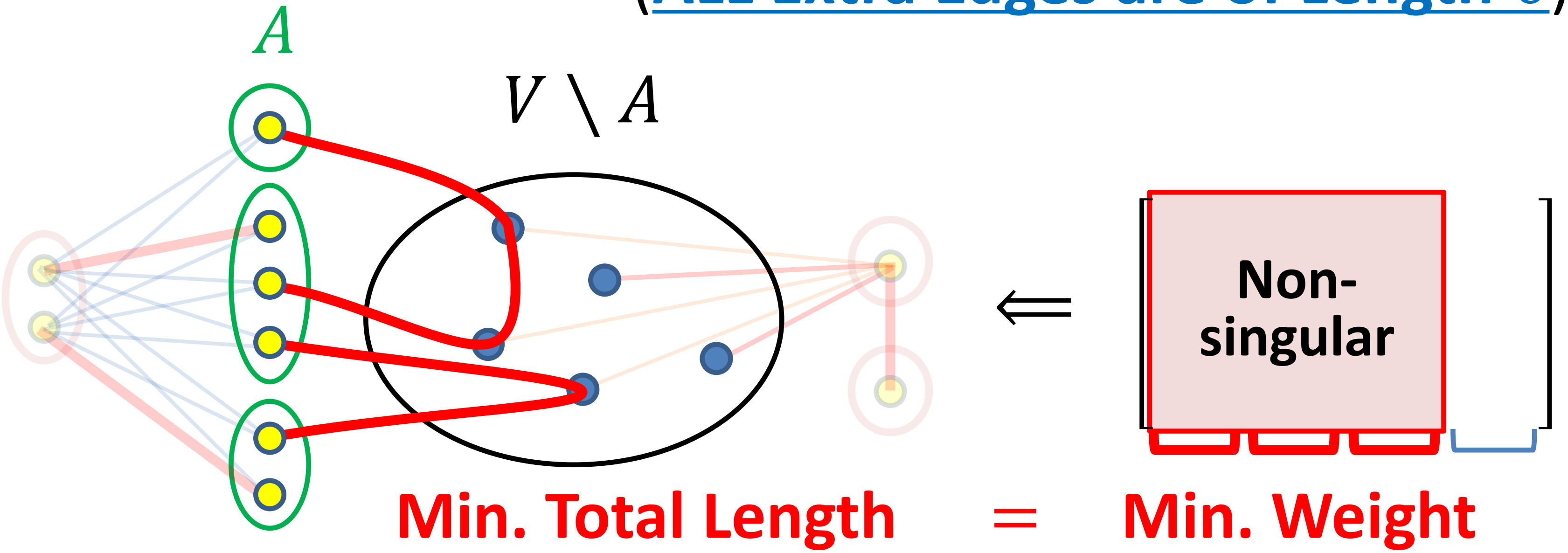

**Min. Total Length of $\mathcal{S}$-paths** $=$ **Min. Weight**

# Summary on Disjoint $\mathcal{S}$-paths

Constructing Auxiliary Instance
(by Adding Weight-0 Elements)

- **Shortest Disjoint $\mathcal{S}$-paths Problem** is solved in Polytime via Weighted L.M.P.
- This result can be extended to **Packing Non-zero $A$-paths in Group-Labeled Graphs** under some Group Representability Condition [Y. 2016]

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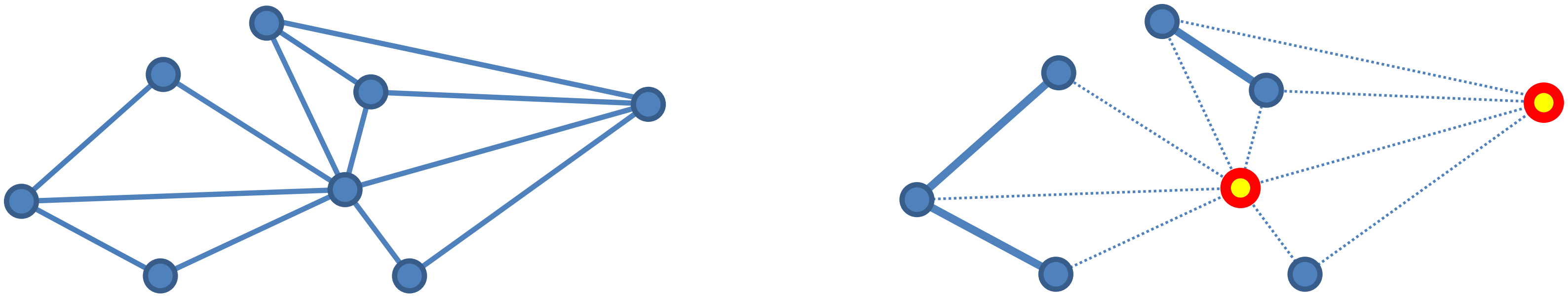
# Feedback Vertex Set Problem

**Given** $G = (V, E)$: Undirected Graph

**Find** **Feedback Vertex Set of Minimum Cardinality**

$X \subseteq V$ s.t. $G - X$ is a **Forest**



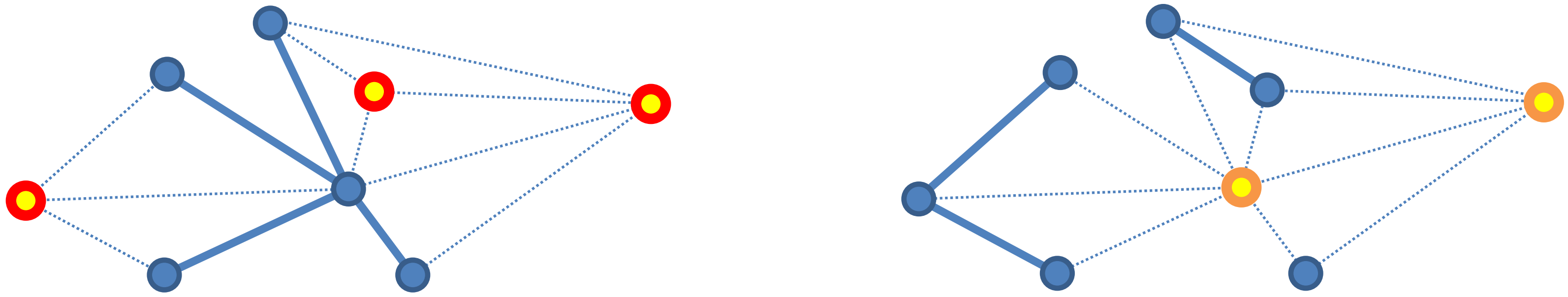
# Minimum-Weight F.V.S. Problem

**Given** $G = (V, E)$: Undirected Graph

$w: V \to \mathbf{R}_{\geq 0}$ **Weight on Vertices**

**Find** **Feedback Vertex Set of Minimum Weight**

**Ex.** $w \equiv d_G$ (Degree of Vertices)



$$3 + 3 + 4 = 10 < 11 = 7 + 4$$

# Minimum-Weight F.V.S. Problem

**Given** $G = (V, E)$: Undirected Graph, $w: V \to \mathbf{R}_{\geq 0}$

**Find** **Feedback Vertex Set of Minimum Weight**

- **NP-Hard** even when
  - $w \equiv \mathbf{1}$ (**Unweighted**), and
  - $G$ is **Planar** with $d_G \leq \mathbf{4}$

[Garey–Johnson 1977]

- **Polytime via L.M.P.** when $w \equiv \mathbf{1}$ and $d_G \leq \mathbf{3}$ (**Subcubic**)

[Ueno–Kajitani–Gotoh 1988]

- Polytime 2-Approximation in General

[Bafna–Berman–Fujito 1999]

# F.V.S. in (Sub)Cubic Graphs → L.M.P.

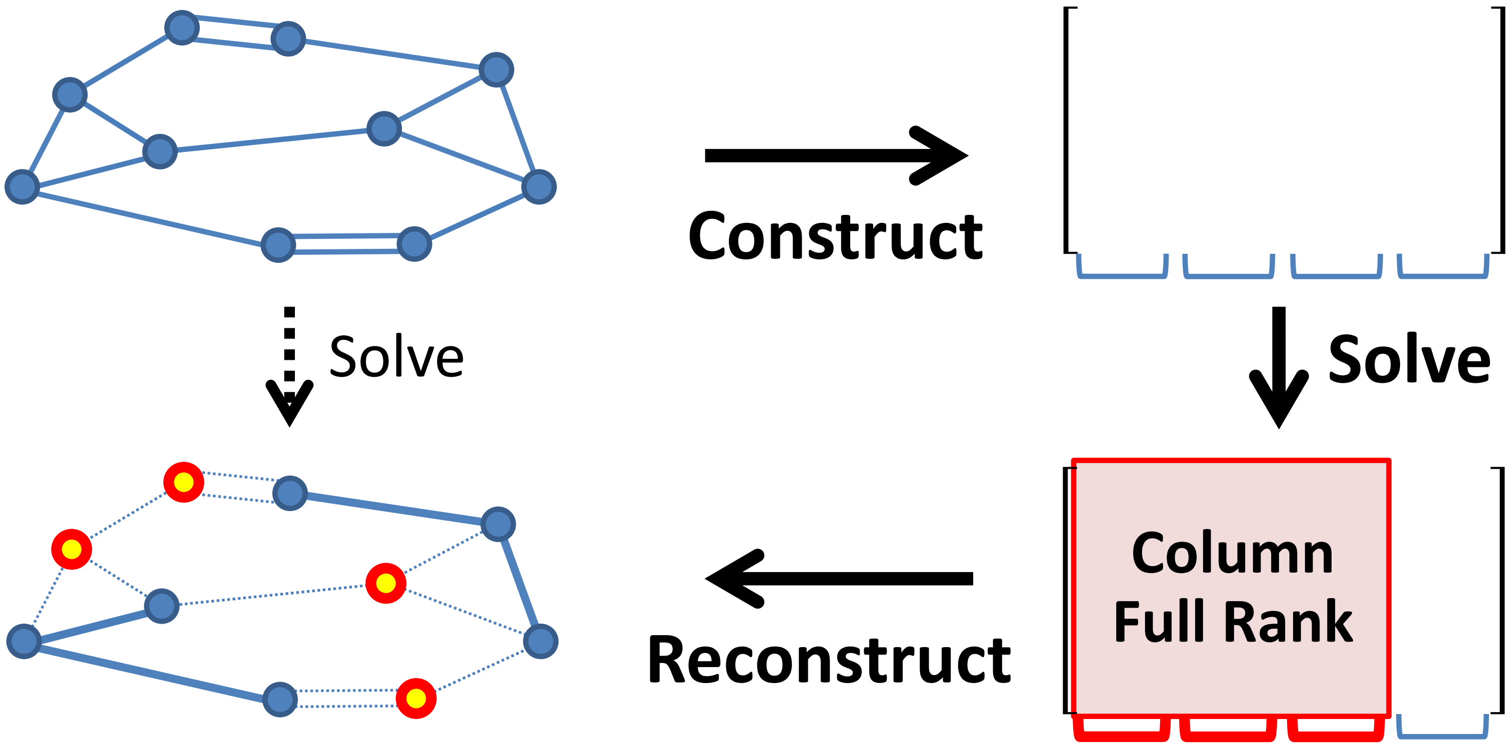
[Ueno–Kajitani–Gotoh 1988]

$d_G \equiv \mathbf{3}$

F.V.S. in **Cubic** Graphs

**Reduce**

Linear Matroid Parity



**Construct**

Solve

**Solve**

**Column Full Rank**

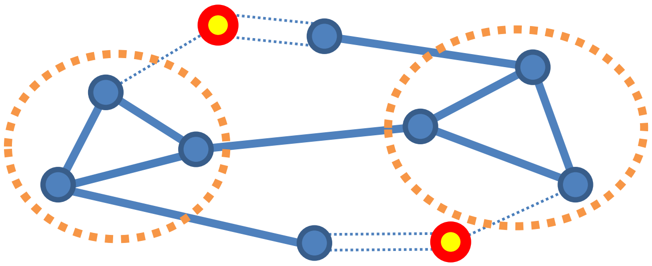
**Reconstruct**

# F.V.S. in (Sub)Cubic Graphs → L.M.P.

[Ueno–Kajitani–Gotoh 1988]

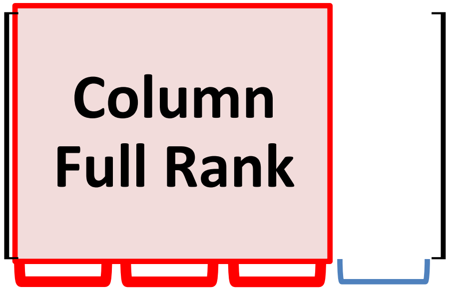
**Thm.** $\forall G=(V,E)$: Cubic, $\exists$ L.M.P. Instance s.t.

- The **Line** set is the **Vertex** set $V$
- If $Y \subseteq V$ is **Optimal**, then
  - each 2-(edge)-conn. comp. of $G-Y$ is a **Cycle**,
  - $\min\{\,|X| \mid X\text{: F.V.S. in } G\,\} = |Y| + \#(\textbf{Cycles} \text{ in } G-Y)$



$\Longleftarrow$

**Max. Card.**



**Column Full Rank**

# F.V.S. in (Sub)Cubic Graphs → L.M.P.

[Ueno–Kajitani–Gotoh 1988]

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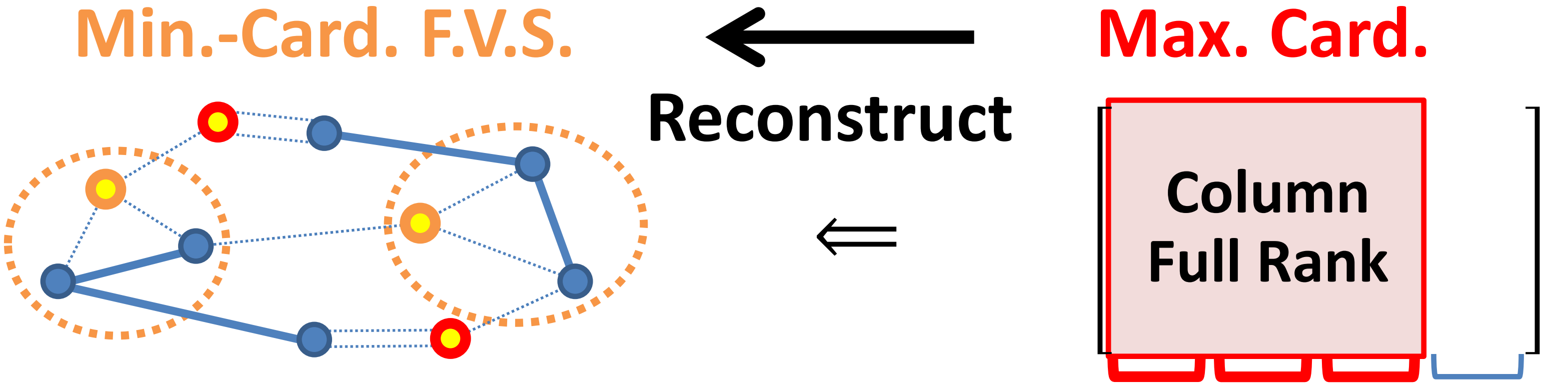

**Min.-Card. F.V.S.**

**Reconstruct**

$\Leftarrow$

**Max. Card.**

**Column Full Rank**

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# Alternative Characterization

[Ueno–Kajitani–Gotoh 1988]

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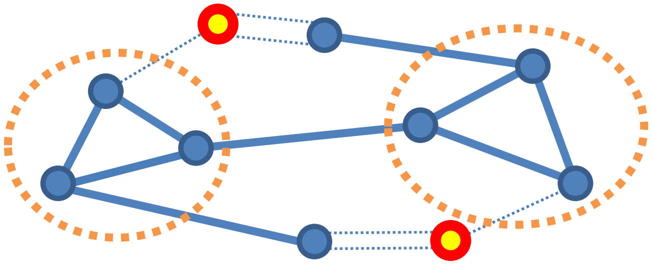
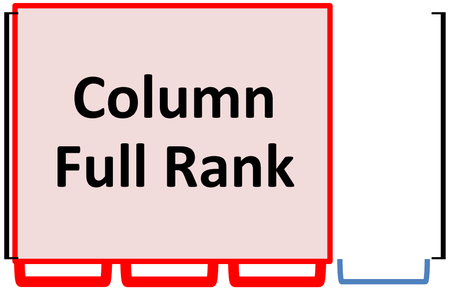


$\Longleftarrow$

**Max. Card.**



**Column Full Rank**

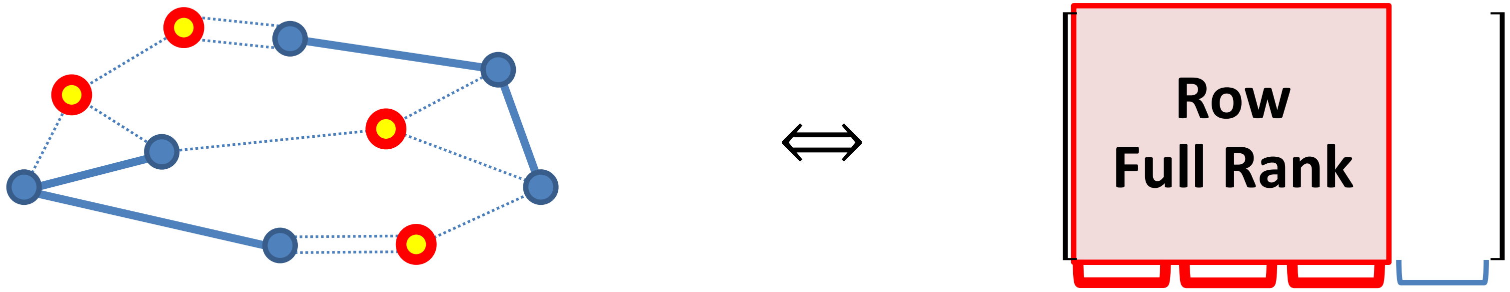
# Alternative Characterization

[Ueno–Kajitani–Gotoh 1988]

**Obs.** $\forall G = (V, E)$: Cubic, $\exists$ L.M.P. Instance s.t.

- The **Line** set is the **Vertex** set $V$
- $X \subseteq V$ is a **F.V.S.** in $G$ $\iff$ $X$ is a **Spanning** Line Subset

Contains a Base (NOT necessarily Parity Base)



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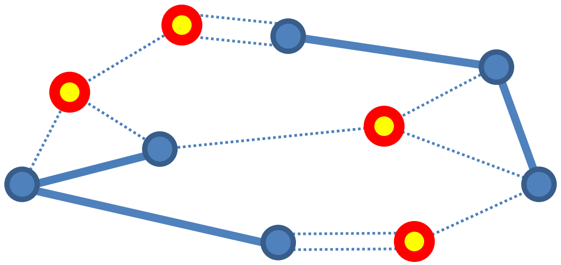
[Ueno–Kajitani–Gotoh 1988]

**Obs.** $\forall G = (V, E)$: Cubic, ∃ L.M.P. Instance s.t.

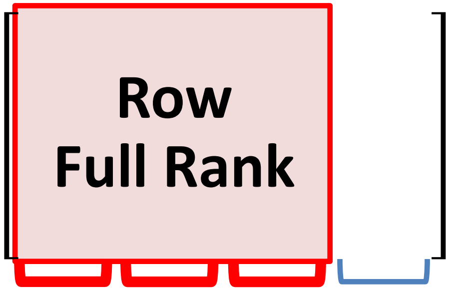
- The **Line** set is the **Vertex** set $V$
- $X \subseteq V$ is a **F.V.S.** in $G$ $\iff$ $X$ is a **Spanning** Line Subset

Contains a Base (NOT necessarily Parity Base)

$\iff$ $V - X$ is **Independent** in the **Dual Matroid**



$\iff$



# Alternative Characterization

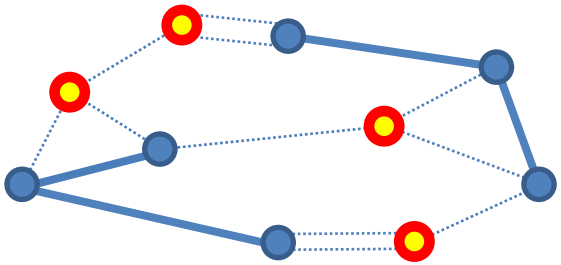
[Ueno–Kajitani–Gotoh 1988]

**Obs.** $\forall G = (V, E)$: Cubic, $\exists$ L.M.P. Instance s.t.

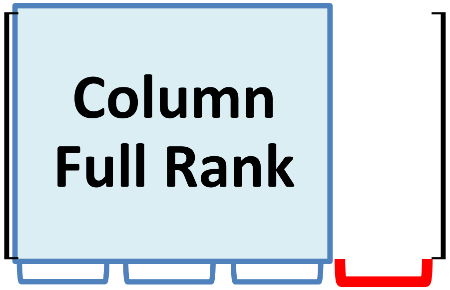
- The **Line** set is the **Vertex** set $V$
- $X \subseteq V$ is a **F.V.S.** in $G$ $\Leftrightarrow$ $V - X$ is **Feasible**

**Fact.** **Dual** of $\mathbf{F}$**-representable** Matroid is $\mathbf{F}$**-representable**

**Min.-Weight. F.V.S.** $\Longleftarrow$ **Max. Weight**



$\Longleftrightarrow$

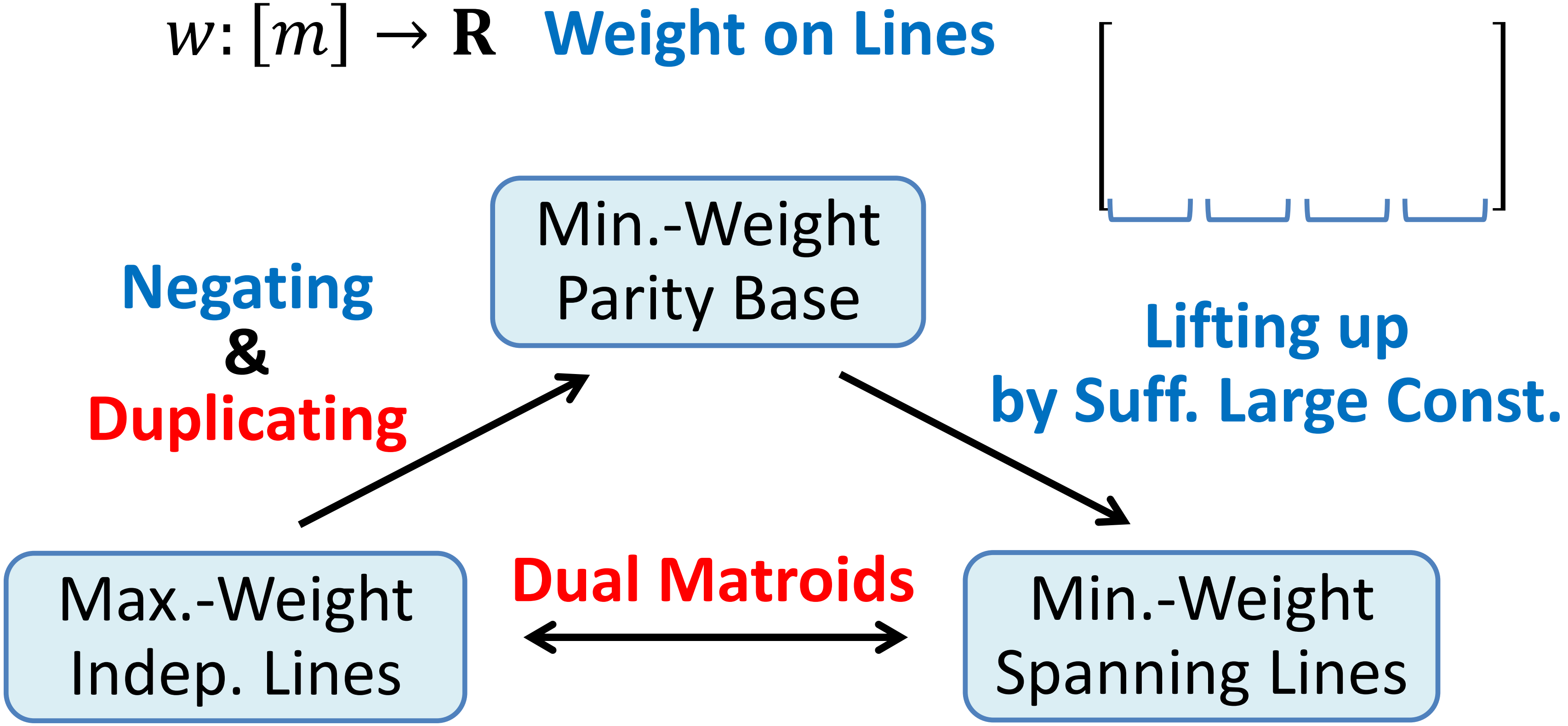


**Column Full Rank**

# Equivalent Formulations of Weighted L.M.P.

**Given** $Z \in \mathbf{F}^{r \times 2m}$: **Matrix with Lines** (Pairing of Columns)

$w : [m] \to \mathbf{R}$ **Weight on Lines**



# Equivalent Formulations of Weighted L.M.P.

**Given** $Z \in \mathbf{F}^{r \times 2m}$: **Matrix with Lines** (Pairing of Columns)

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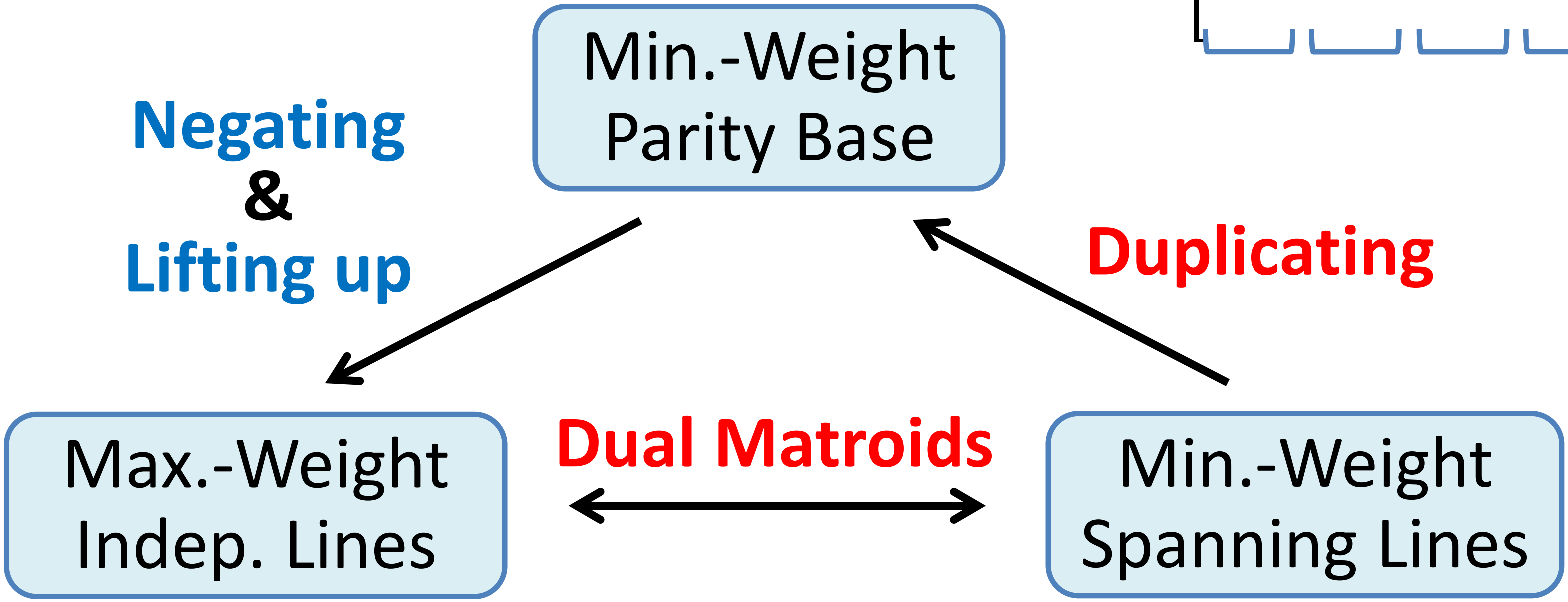


# Summary on F.V.S. in (Sub)Cubic Graphs

Using Alternative Formulations
of Weighted Linear Matroid Parity

- **Minimum-Weight F.V.S. Problem in Subcubic Graphs** is solved in Polytime via Weighted L.M.P.
- In fact, our reduction can be regarded as **Finding Maximum Forests in $3$-Uniform Hypergraphs**, which reduces to L.M.P. in Unweighted case [Lovász 1980]

# Outline

- Preliminaries
- Disjoint $\mathcal{S}$-paths
  - Background
  - Extension Trick
- Feedback Vertex Sets in (Sub)Cubic Graphs
  - Background
  - Extension Trick
- Conclusion

# Conclusion

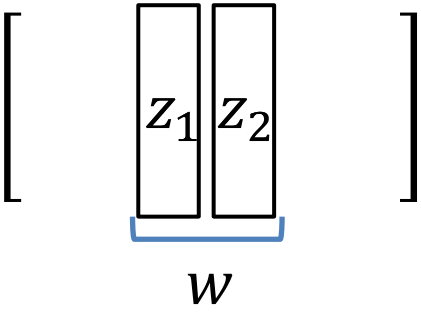
- **Weighted L.M.P.** is Very Powerful to show **Tractability**

- Two General Strategies
  to Extend Applications of L.M.P. to Weighted Situations
  - Construct Auxiliary Instance (with Weight-0 Elements)
  - Use Alternative Formulations of Weighted L.M.P.

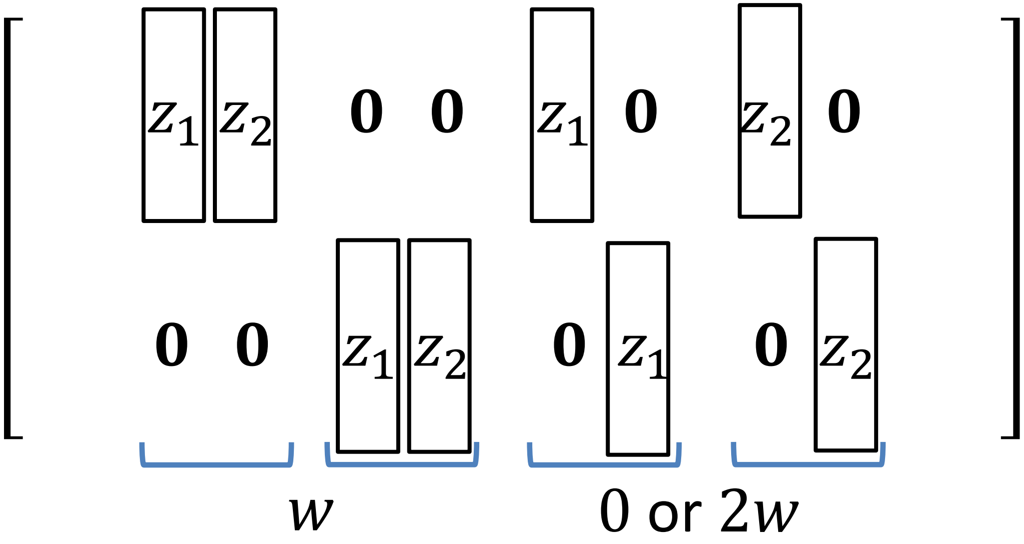
- Some Tricky or Other-type Applications??
  - Like, e.g., **Shortest Path** & $T$**-join** → Weighted Matching?
  - Derive **Min-Max Duality** or **Polyhedral** Property?

# Appendix

# Duplicating of L.M.P. Instance



Original



Duplicated Instance

**Feasible** $\times 2$ $\Leftarrow$ **Restriction of Parity Base**

**Spanning** $\times 2$ $\Leftarrow$ **Parity Base**