Packing A-paths in Group-Labelled Graphs via Linear Matroid Parity

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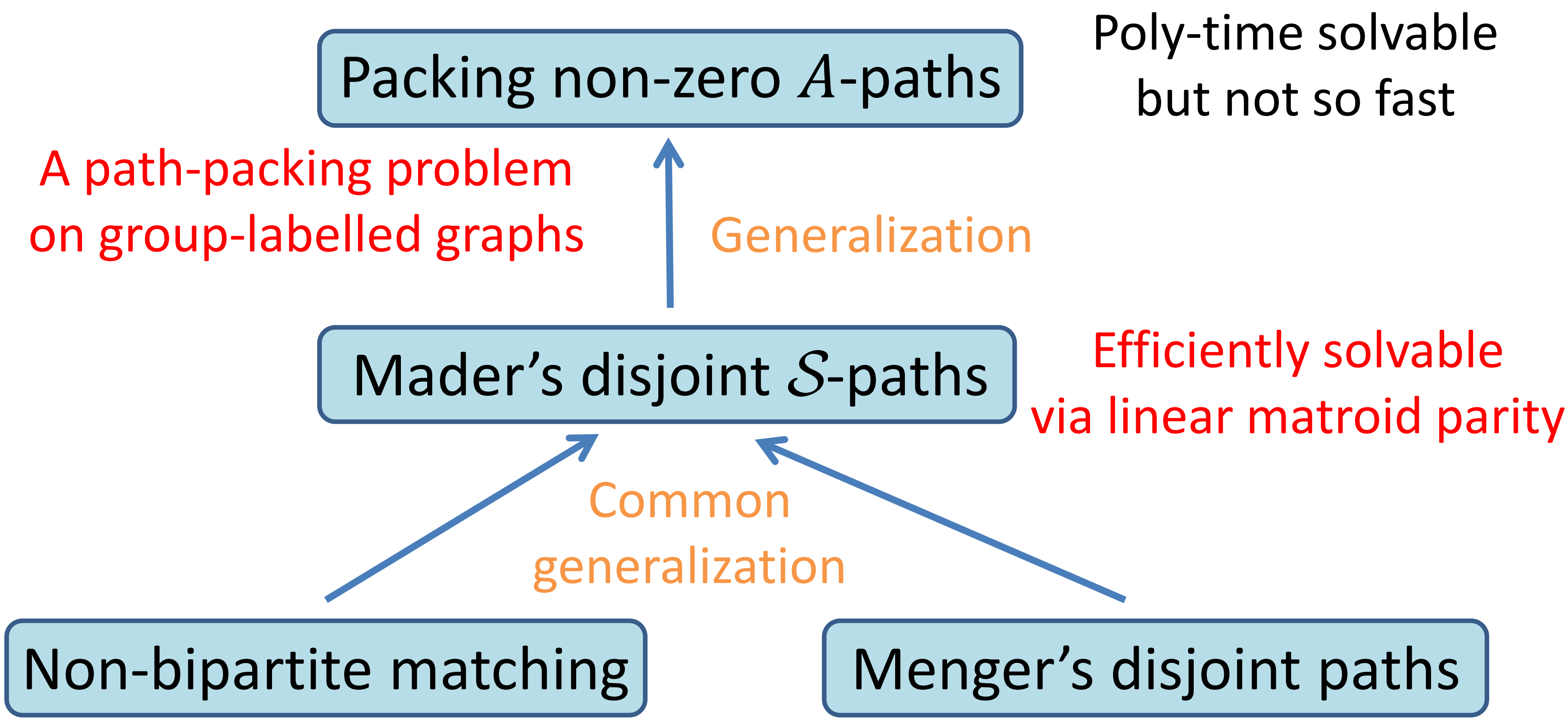
Overview


Poly-time solvable
but not so fast
Packing non-zero A-paths
A path-packing problem
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Generalization
Mader's disjoint $\mathcal{S}$-paths
Efficiently solvable
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Common
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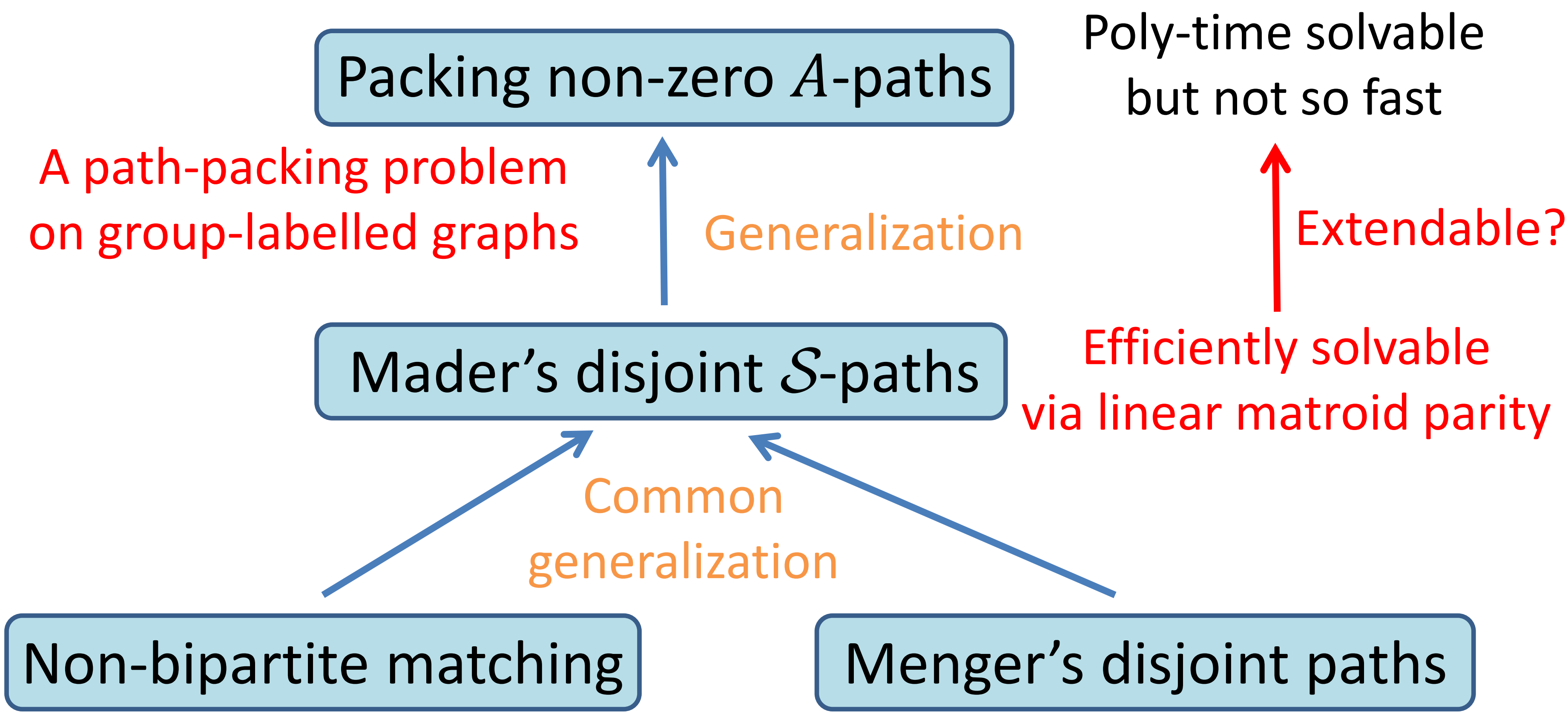
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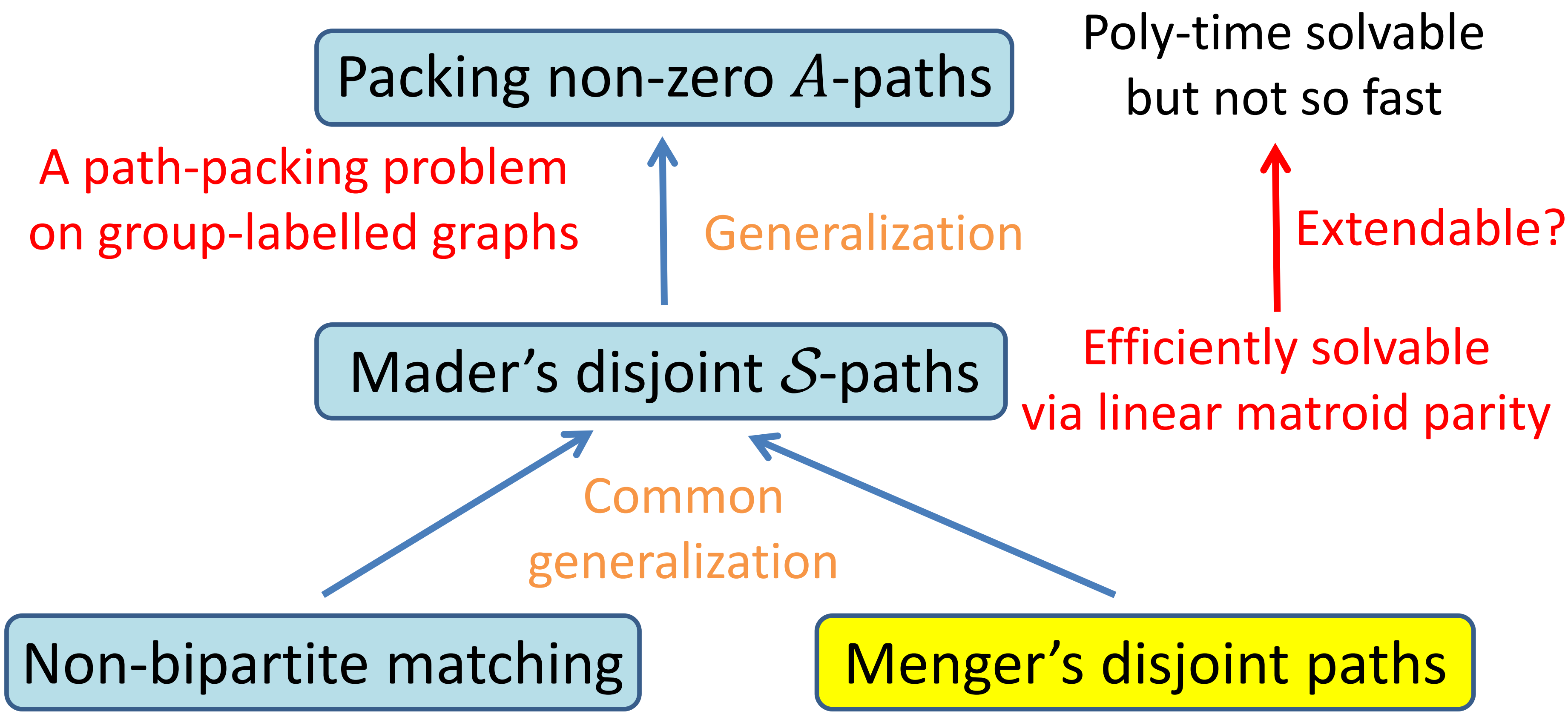
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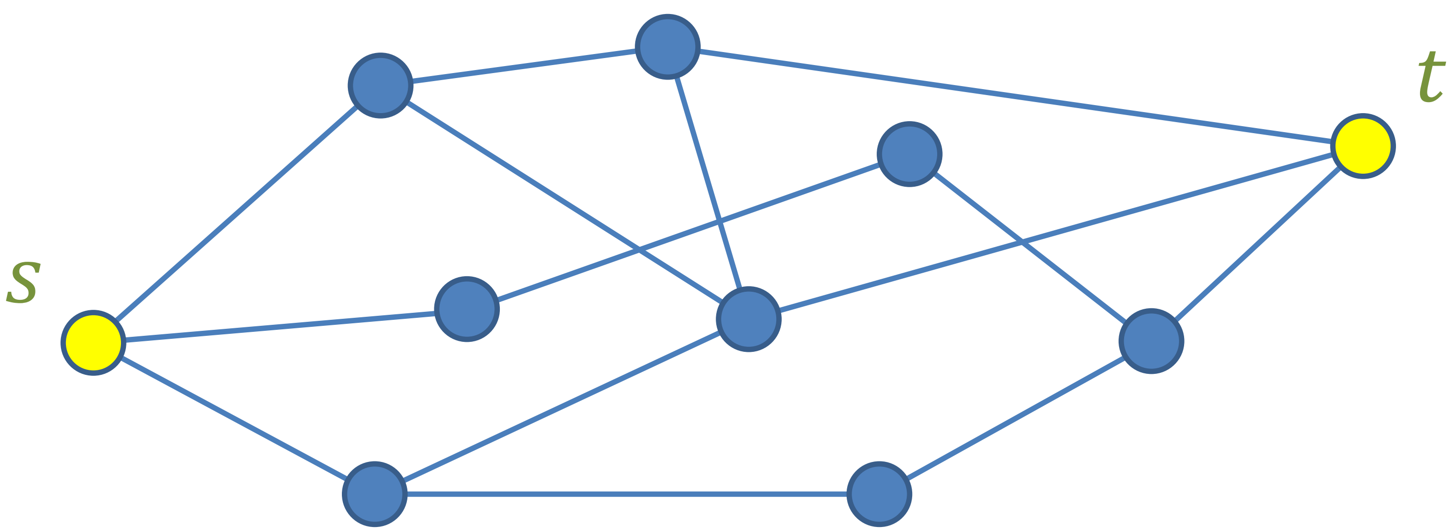
Menger's disjoint paths problem

Input: $G = (V, E)$: undirected graph, $s, t \in V$

Find: a maximum family of (internally) vertex-disjoint paths between s and t in G



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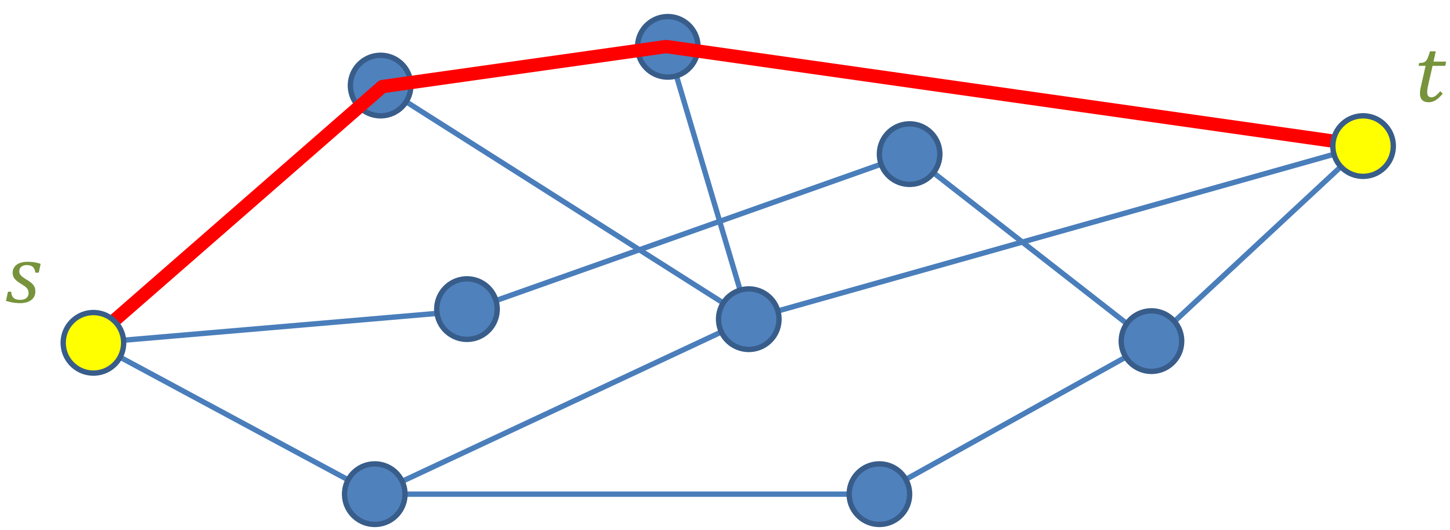

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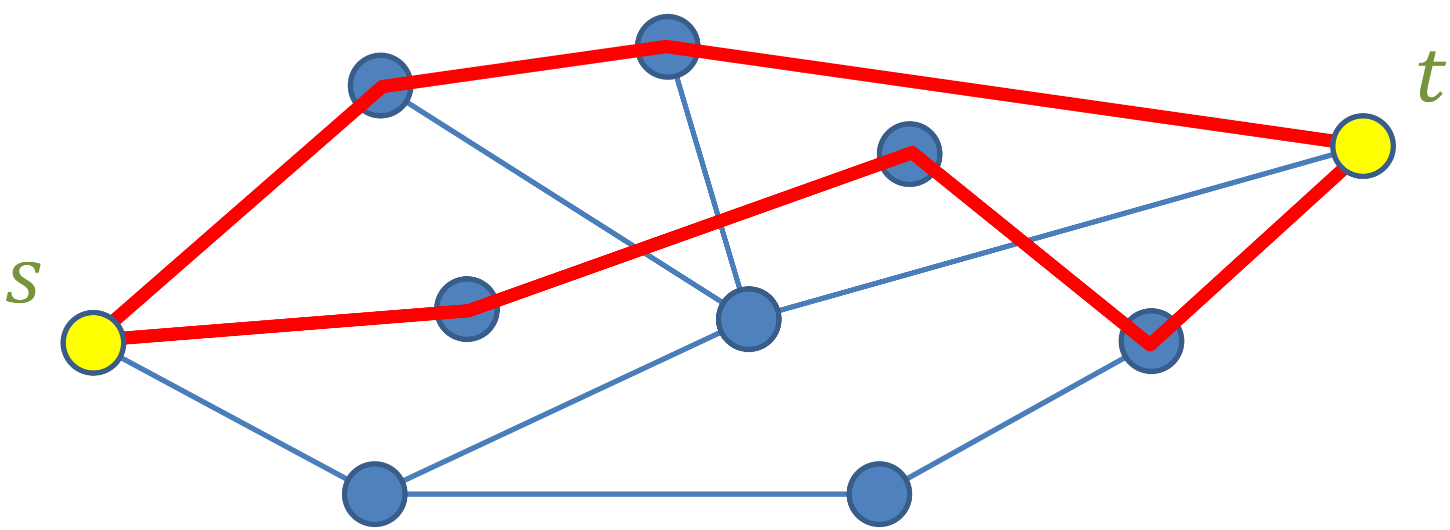

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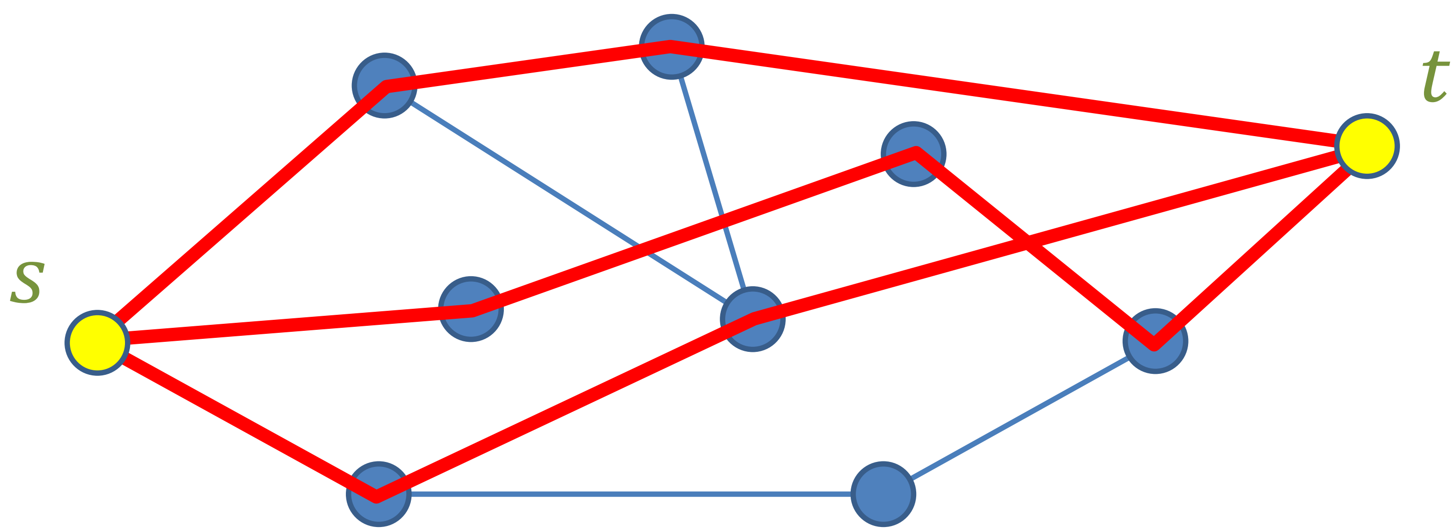

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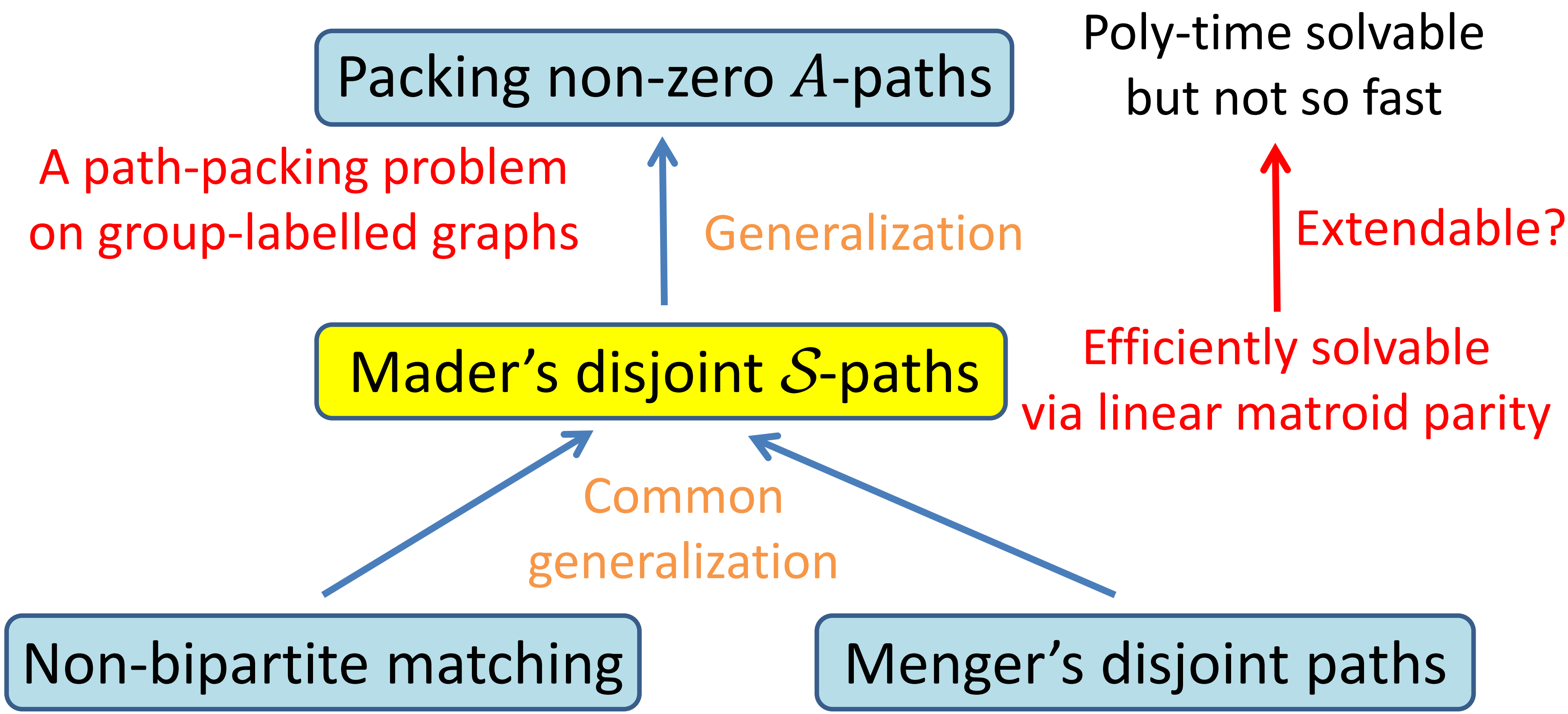

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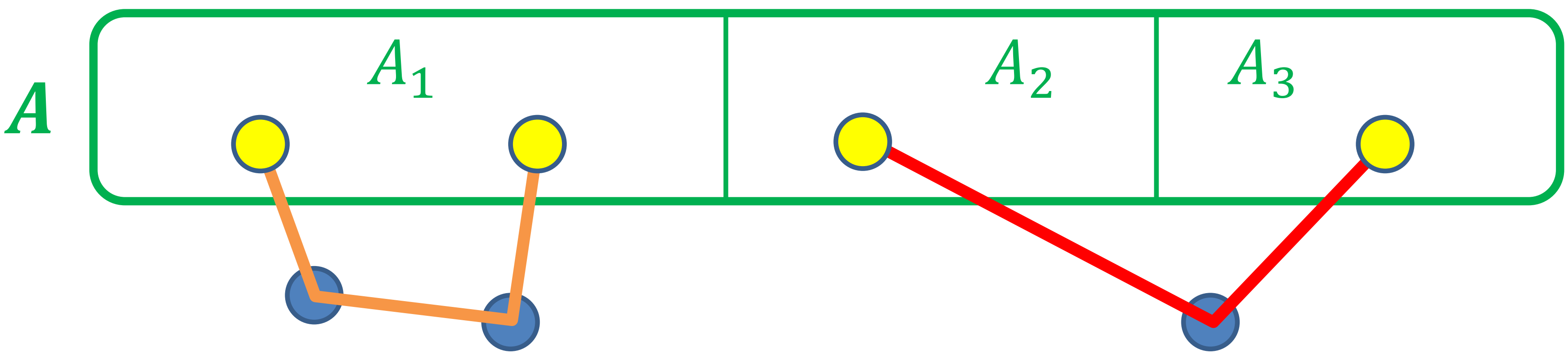
A-paths and $\mathcal{S}$-paths

$G = (V, E)$: undirected graph

$A \subseteq V$: terminal set, $\mathcal{S} = \{A_1, \dots, A_k\}$: partition of A

- An A-path is a path between distinct terminals in A whose inner vertices are not in A.
- An $\mathcal{S}$-path is an A-path between distinct classes in $\mathcal{S}$.



An A-path, NOT an $\mathcal{S}$-path

An $\mathcal{S}$-path

Mader's disjoint $\mathcal{S}$-paths problem

Input: $G = (V, E)$: undirected graph

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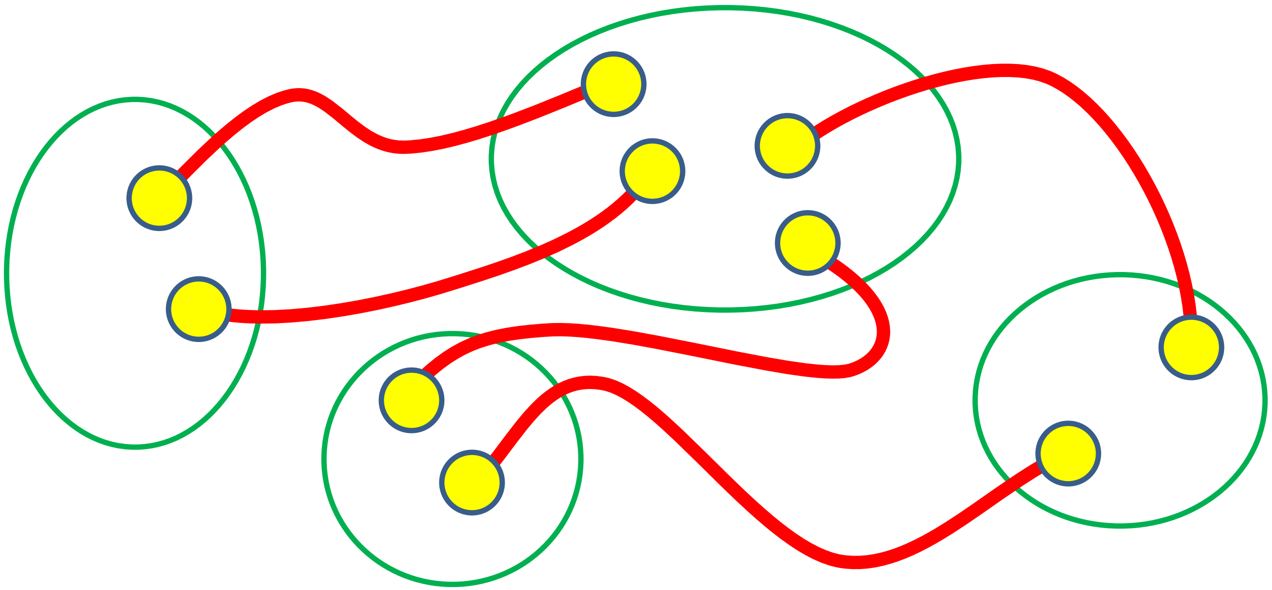
Find: a maximum family of (fully) vertex-disjoint $\mathcal{S}$-paths in G



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Input: $G = (V, E)$: undirected graph

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Find: a maximum family of (fully) vertex-disjoint $\mathcal{S}$-paths in G

- Min-max formula (Mader 1978)
- Reduction to matroid matching (Lovász 1980)

  → Poly-time solvability (one can obtain a "good" matroid)
- Linear representation of the matroid (Schrijver 2003)

  → More efficient solvability (via linear matroid parity)

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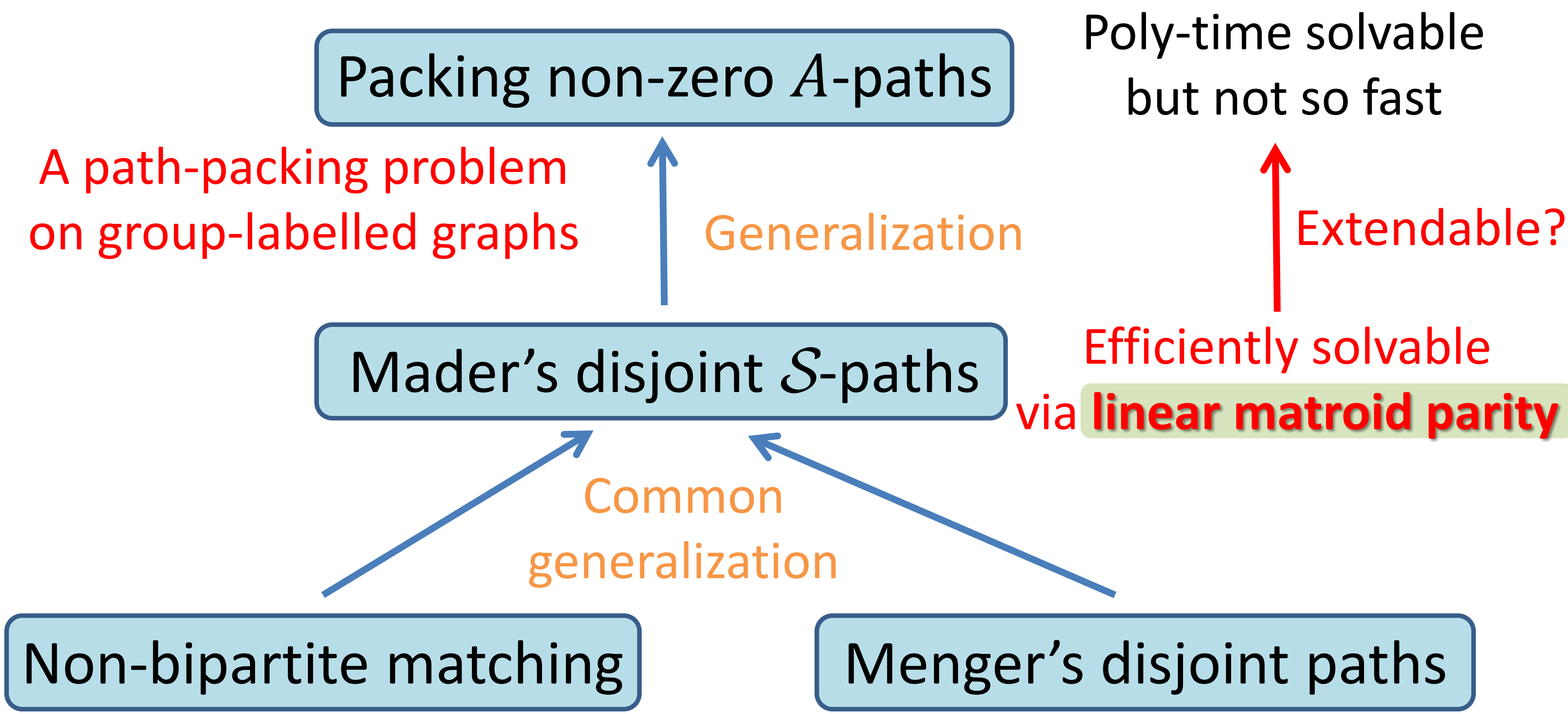

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Linear matroid parity problem

Input: a matrix $Z \in \mathbf{F}^{n \times 2m}$ with pairing of the columns

Find: a maximum family of column-pairs whose union is linearly independent

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

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Linear matroid parity problem

Input: a matrix $Z \in \mathbf{F}^{n \times 2m}$ with pairing of the columns

Find: a maximum family of column-pairs
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- Solvable in $O(m^{17})$ time (Lovász 1981)
- Solvable in $O(mn^3)$ time (Gabow, Stallmann 1986)
- Solvable in $O(mn^2)$ time w.h.p. (Cheung, Law, Leung 2011)

If fast matrix multiplication is used, then, for $\omega \approx 2.376$

- Solvable in $O(mn^{\omega})$ time (Gabow, Stallmann 1986)
- Solvable in $O(mn^{\omega-1})$ time w.h.p. (Cheung et al. 2011)

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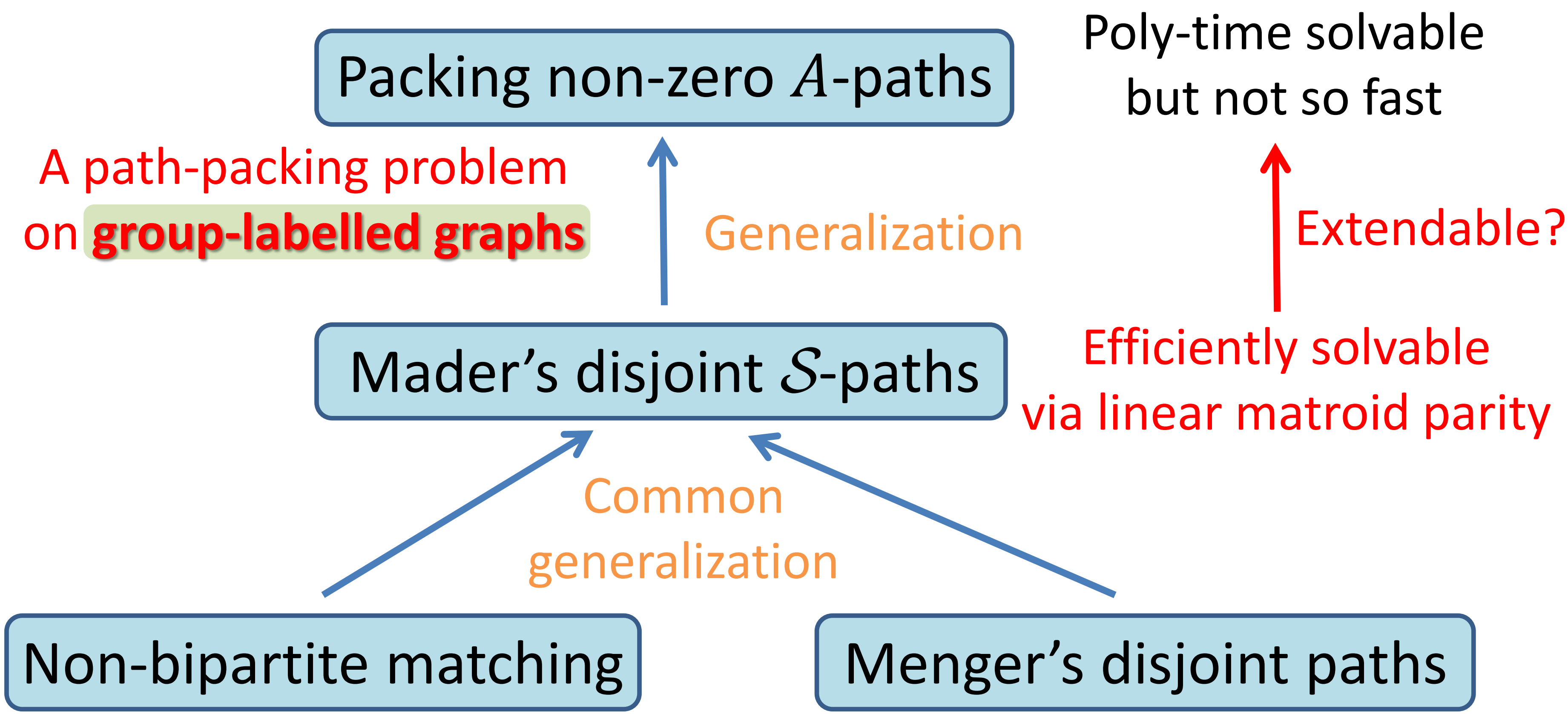

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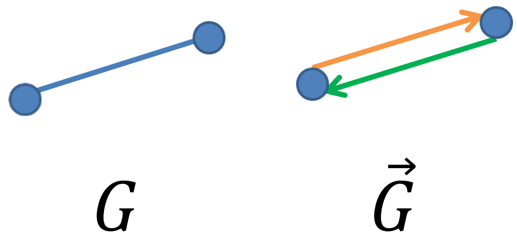
Group-labelled graphs

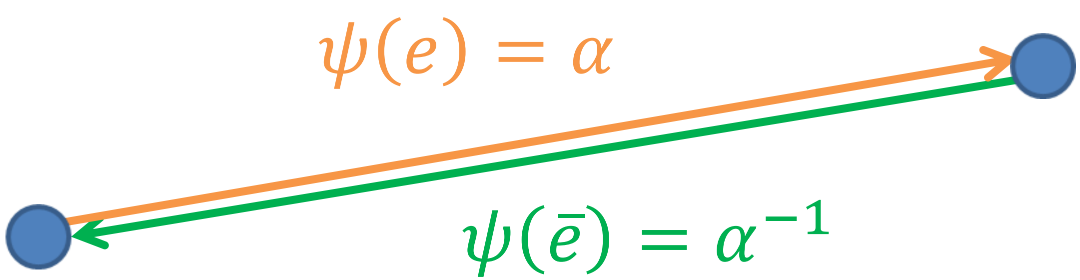
$G = (V, E)$: undirected graph

$\vec{G} = (V, \vec{E})$: two-way orientation of G



Γ: group

$\psi: \vec{E} \to \Gamma$ with $\psi(\bar{e}) = \psi(e)^{-1}$ for each $e \in \vec{E}$



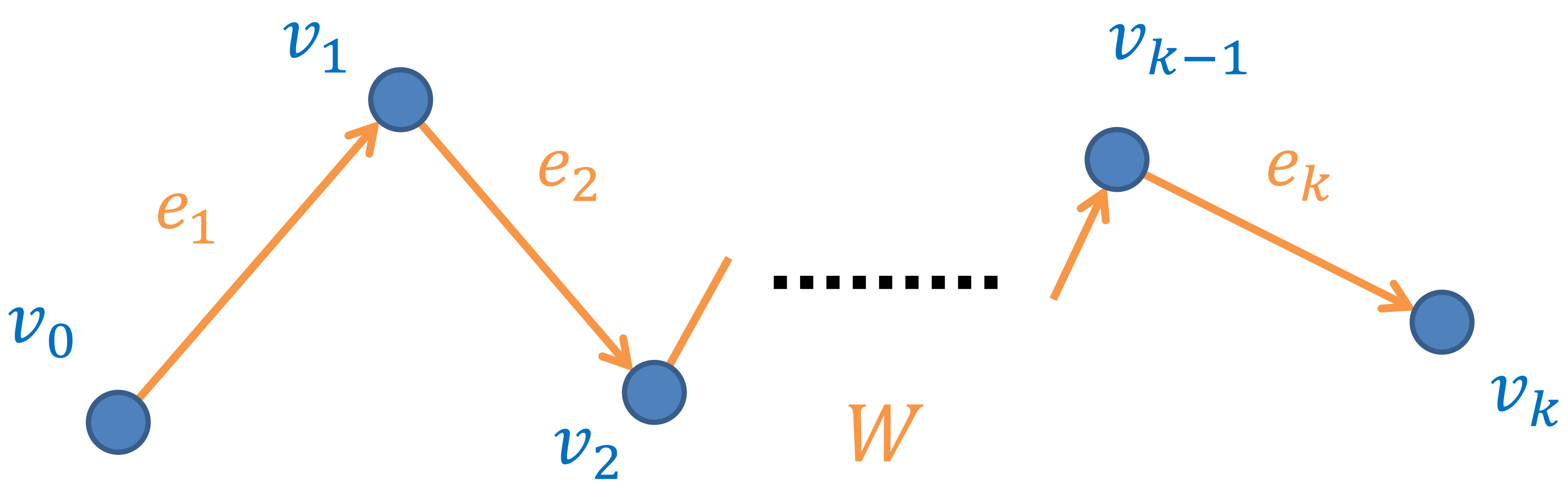
A pair (G, ψ) is called a Γ-labelled graph.

Labels of walks

$\psi: \vec{E} \to \Gamma$ with $\psi(\bar{e}) = \psi(e)^{-1}$ for each $e \in \vec{E}$

The label $\psi(W)$ of a walk $W = (v_0, e_1, v_1, \dots, e_k, v_k)$ is

$$\psi(W) \coloneqq \psi(e_k) \cdots \psi(e_2) \cdot \psi(e_1)\,.$$



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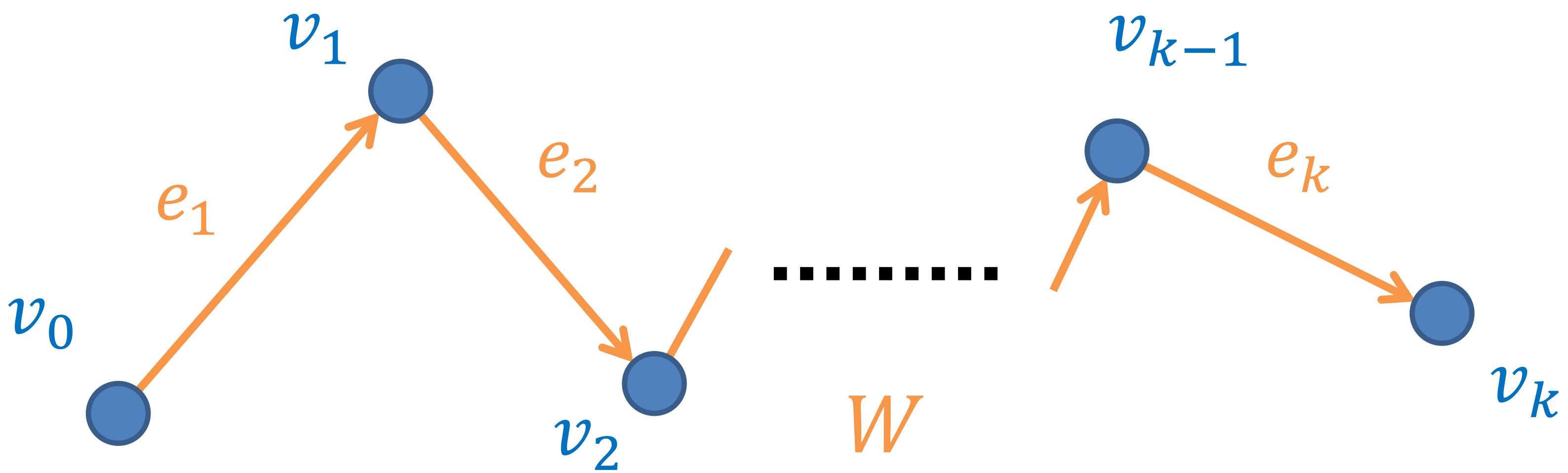
$$\begin{aligned}\psi(\overline{W}) &= \psi(\bar{e}_1) \cdot \psi(\bar{e}_2) \cdots \psi(\bar{e}_k) \\ &= \psi(e_1)^{-1} \cdot \psi(e_2)^{-1} \cdots \psi(e_k)^{-1} = \psi(W)^{-1}\end{aligned}$$



Packing A-paths in group-labelled graphs

- **Non-zero model** (Chudnovsky, Geelen, Gerards, Goddyn, Lohman, Seymour 2006)

  An A-path P can be used for packing $\Leftrightarrow$ $\psi(P) \neq 1_\Gamma$.

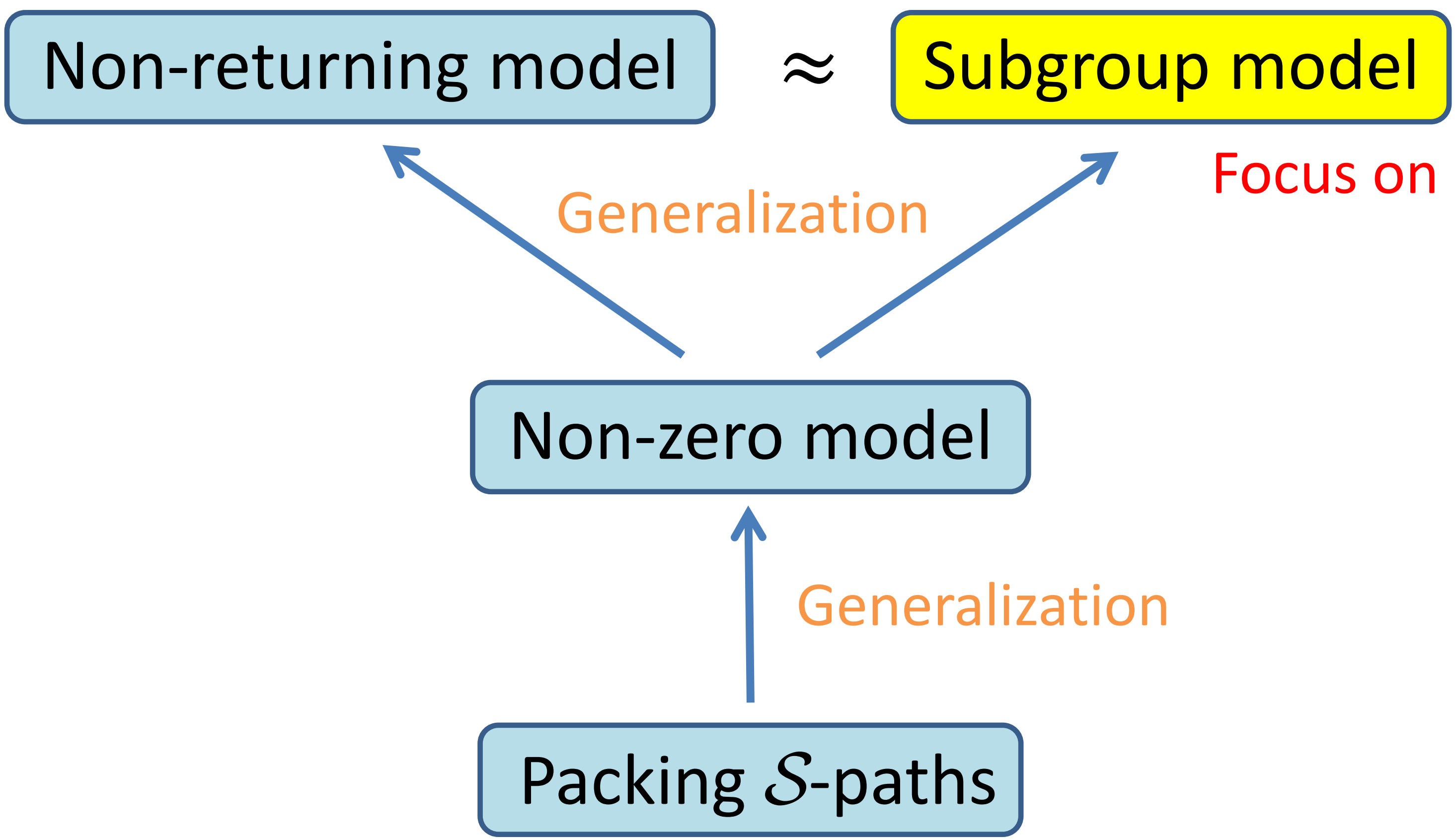
- **Non-returning model** (Pap 2007)

  Γ is a symmetric group S_d, the set of permutations on $\{1, \dots, d\}$.
  An A-path P can be used for packing $\Leftrightarrow$ $\big(\psi(P)\big)(d) \neq d$.

- **Subgroup model** (Pap)

  For a prescribed proper subgroup $\Gamma' \subset \Gamma$,
  an A-path P can be used for packing $\Leftrightarrow$ $\psi(P) \notin \Gamma'$.

Relation among the models

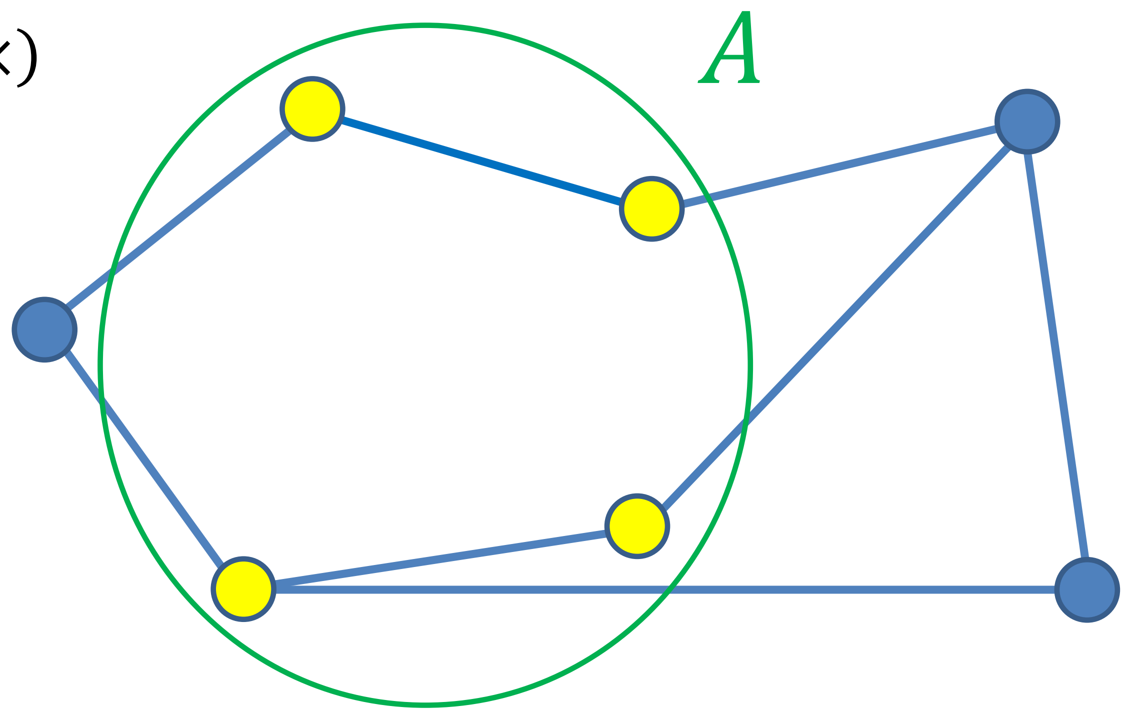


Subgroup model

Γ': proper subgroup of Γ

An A-path P is admissible $\overset{\text{def}}{\Leftrightarrow}$ $\psi(P) \notin \Gamma'$.

$$\Gamma = (\{1, -1\}, \times)$$
$$\Gamma' = \{1\}$$
$$\psi(e) \equiv -1$$



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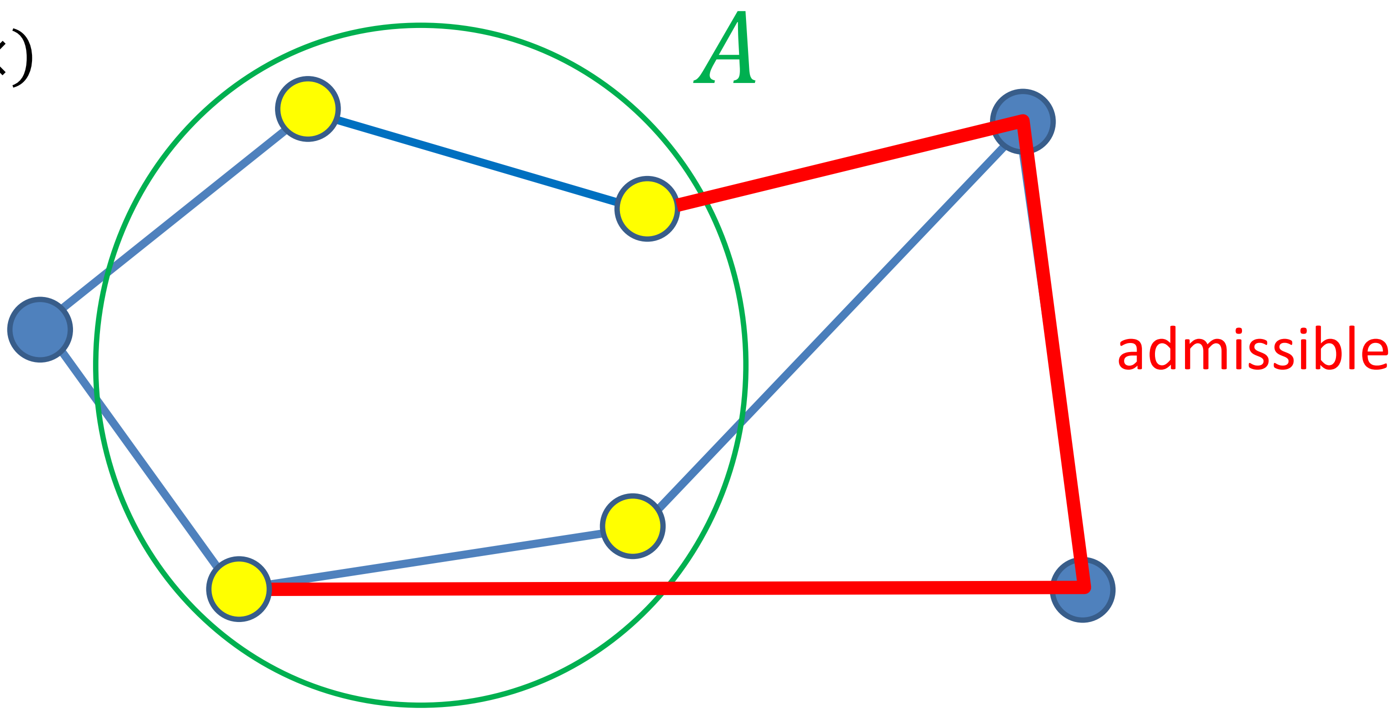

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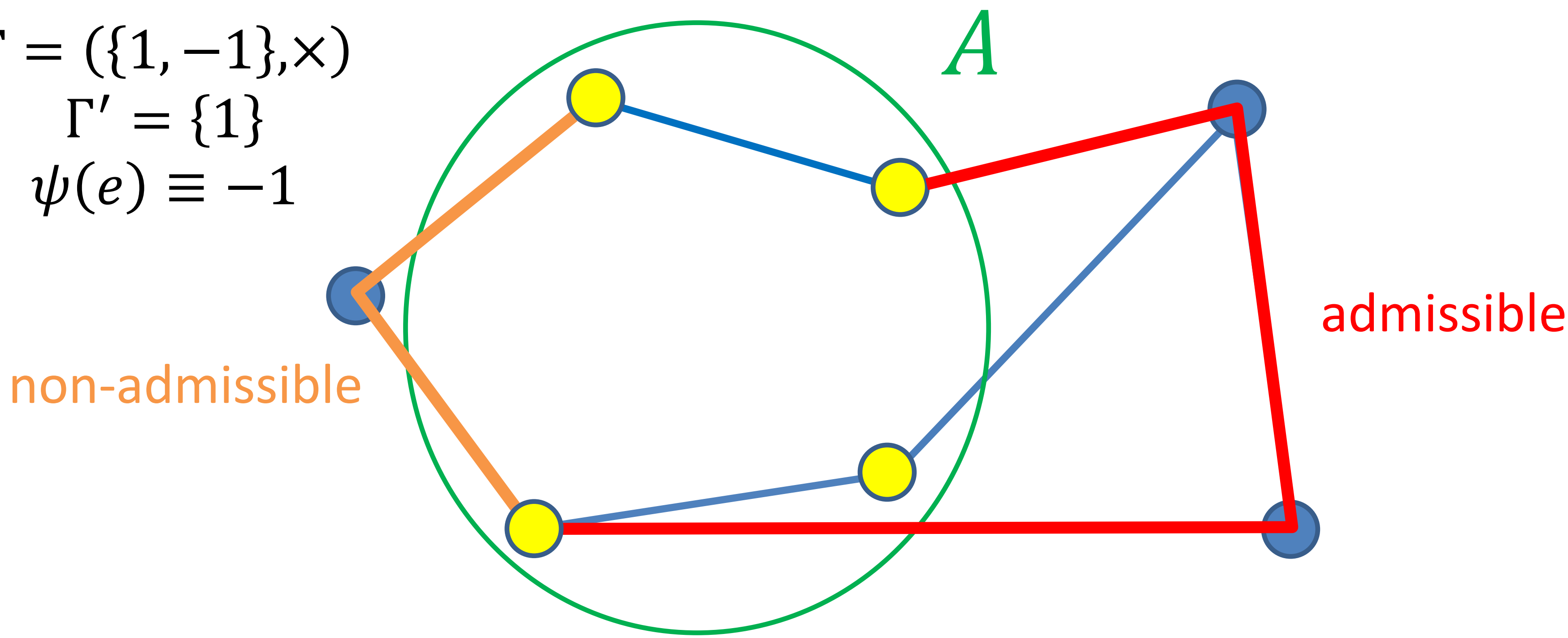

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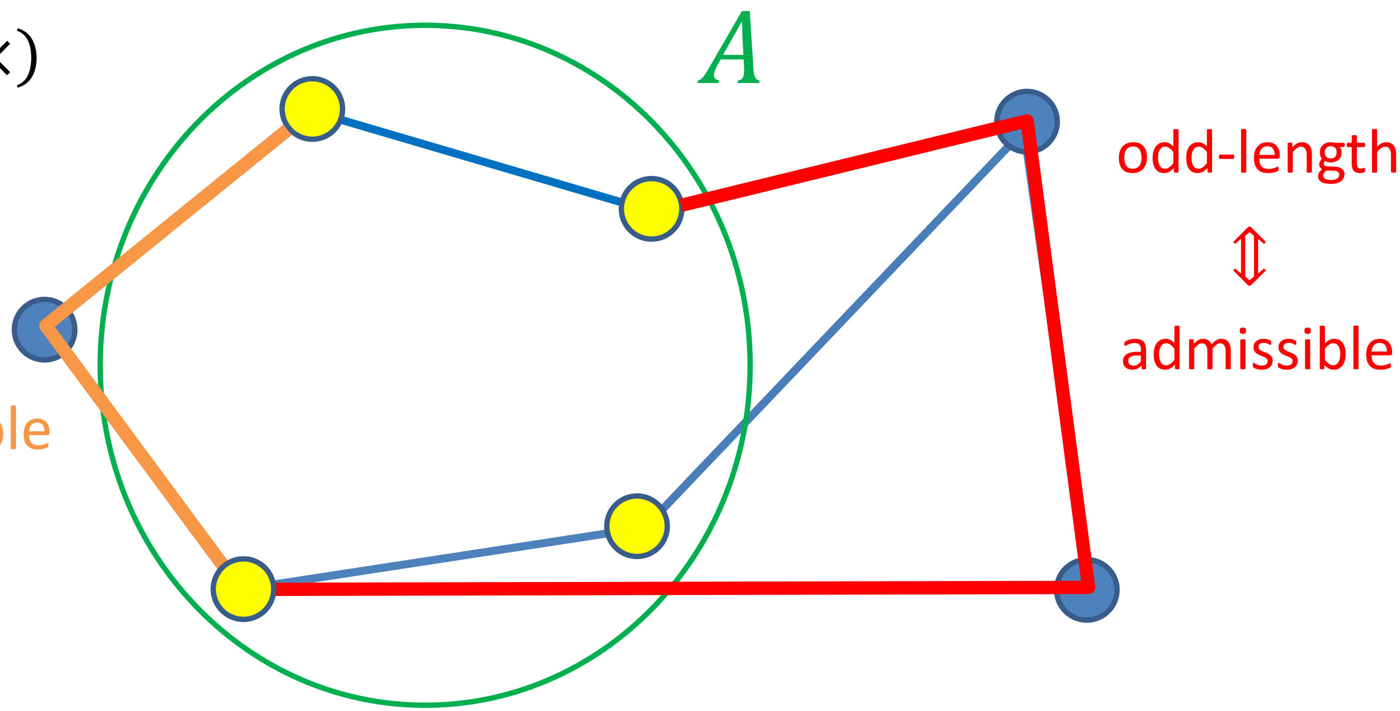

Subgroup model of packing A-paths

Input: (G, ψ): Γ-labelled graph
$A \subseteq V(G)$: terminal set, Γ': proper subgroup of Γ

Find: a maximum family of
(fully) vertex-disjoint admissible A-paths in (G, ψ)

- Min-max formula (Pap 2007)
- No explicitly polynomial-bounded algorithm was known…

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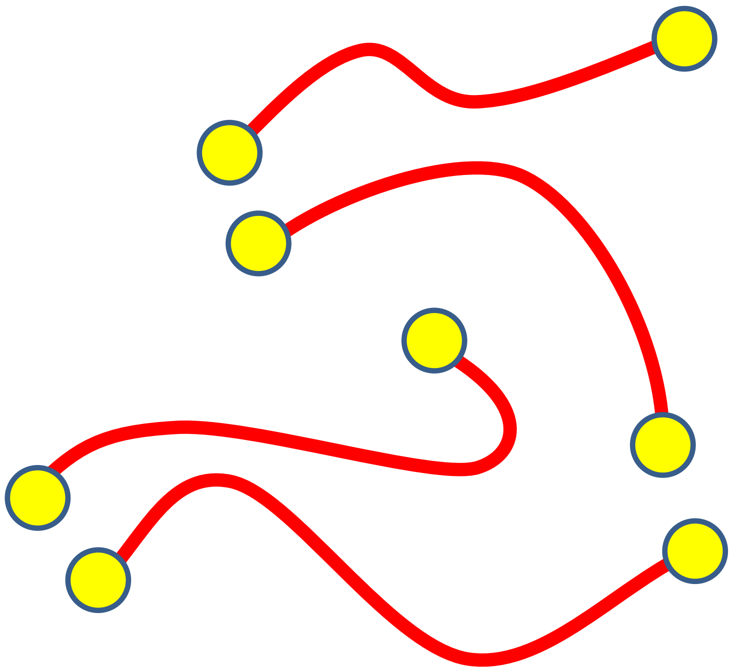
  → Extension of algorithm for the non-zero model
  → Reduction to linear matroid parity

Algorithms for subgroup model

- Extension of algorithm for the non-zero model
  - Extend the combinatorial algorithm of Chudnovsky, Cunningham, Geelen (2008)
  - Always applicable
  - Not so fast, $O(|V(G)|^5)$ time
- Reduction to linear matroid parity
  - Extend the linear representation of Schrijver (2003)
  - Not always applicable
  - Faster, $O(|V(G)|^\omega)$ time w.h.p. for simple graphs, for example, where $\omega \approx 2.376$ is the matrix multiplication exponent

What is desired for reduction?

Subgroup model — reduce → Linear matroid parity



Vertex-disjoint admissible A-paths ← uniquely — Linearly independent column-pairs

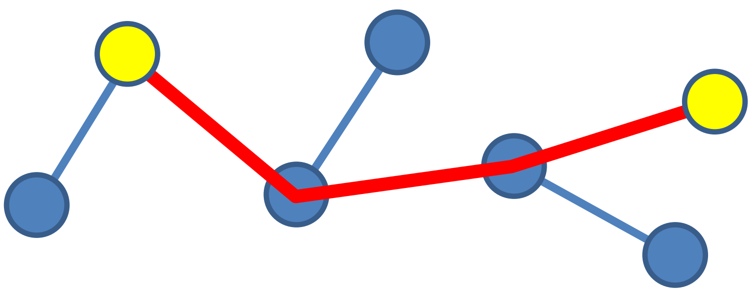
What is desired for reduction?

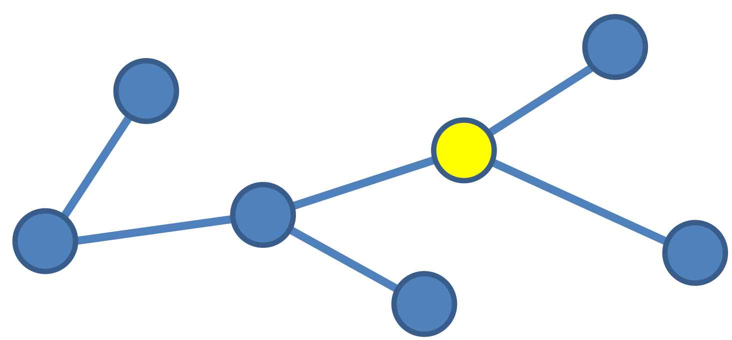
Subgroup model $\xrightarrow{\text{reduce}}$ Linear matroid parity

Vertex-disjoint admissible A-paths $\xleftarrow{\text{uniquely}}$ Linearly independent column-pairs

$\Uparrow$

- Edge $\leftrightarrow$ Column-pair (Edge set $\leftrightarrow$ Column-pairs)
- Each connected component formed by a feasible edge set contains at most one A-path, which is admissible.





Sufficient condition for reduction

Γ: group, Γ': proper subgroup of Γ, $\mathbf{F}$: field

n: positive integer, I_n: $n \times n$ identity matrix

- $\mathrm{GL}(n, \mathbf{F})$: set of all nonsingular $n \times n$ matrices over $\mathbf{F}$
- $\mathrm{PGL}(n, \mathbf{F}) \coloneqq \mathrm{GL}(n, \mathbf{F}) / \{ kI_n \mid k \in \mathbf{F} \}$

Main theorem (one direction)

$^\exists \rho: \Gamma \to \mathrm{PGL}(2, \mathbf{F})$ homomorphic, $^\exists Y$: 1-dim. subspace of $\mathbf{F}^2$

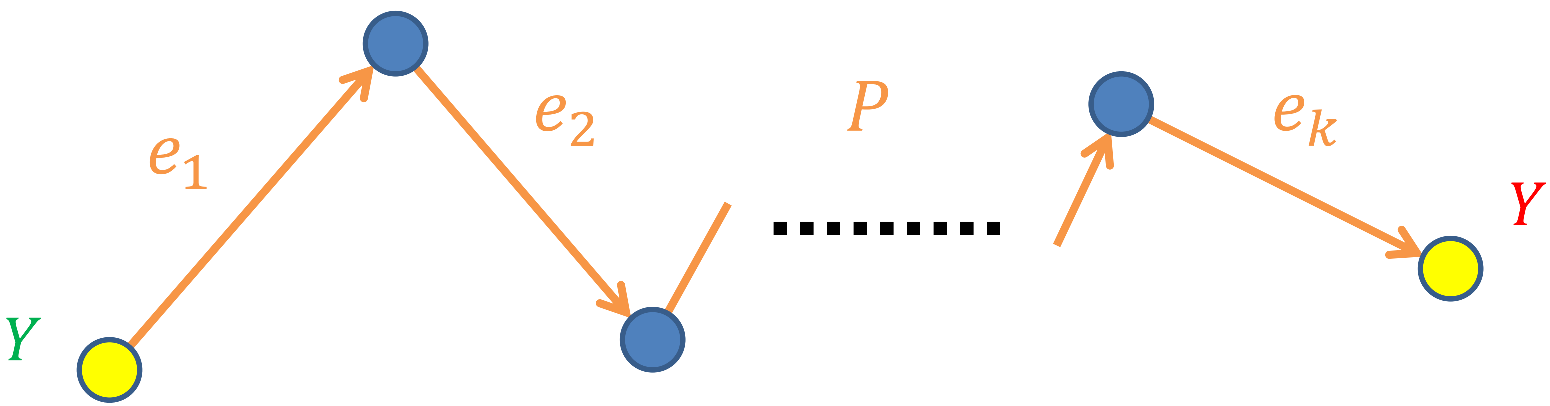
s.t. $\Gamma' = \{ \alpha \in \Gamma \mid \rho(\alpha) Y = Y \}$

$\Rightarrow$ Subgroup model reduces to linear matroid parity.

Idea of reduction

$\rho : \Gamma \to \mathrm{PGL}(2, \mathbf{F})$ homomorphic, Y: 1-dim. subspace of $\mathbf{F}^2$
s.t. $\Gamma' = \{ \alpha \in \Gamma \mid \rho(\alpha) Y = Y \}$ ($\mathbf{F}$: field)

- Each terminal in A is associated with Y.
- Each edge acts and carries 1-dim. subspaces.



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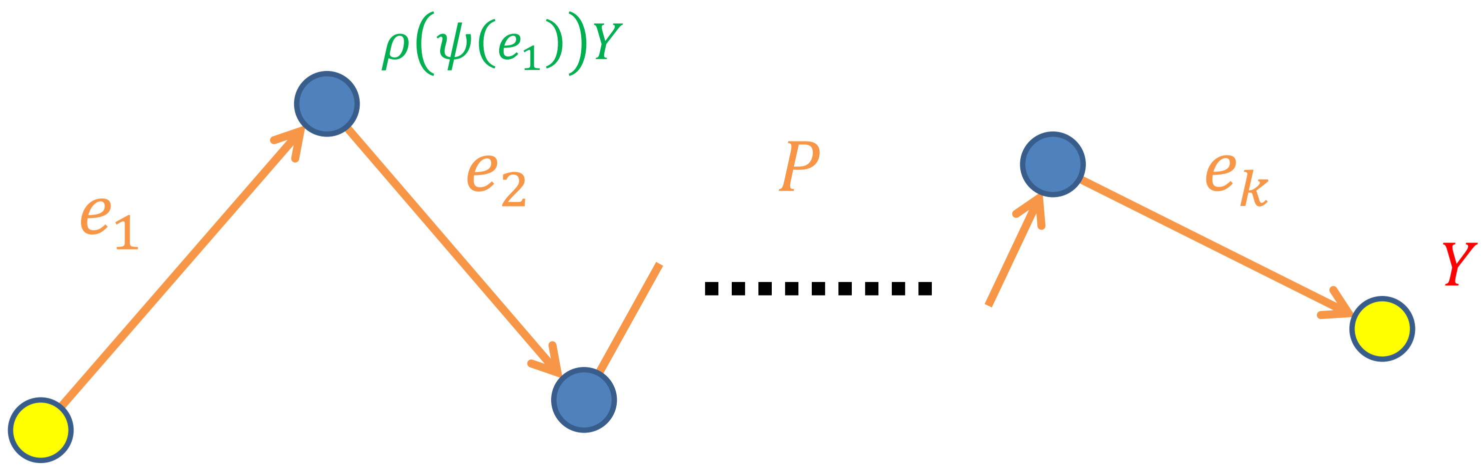

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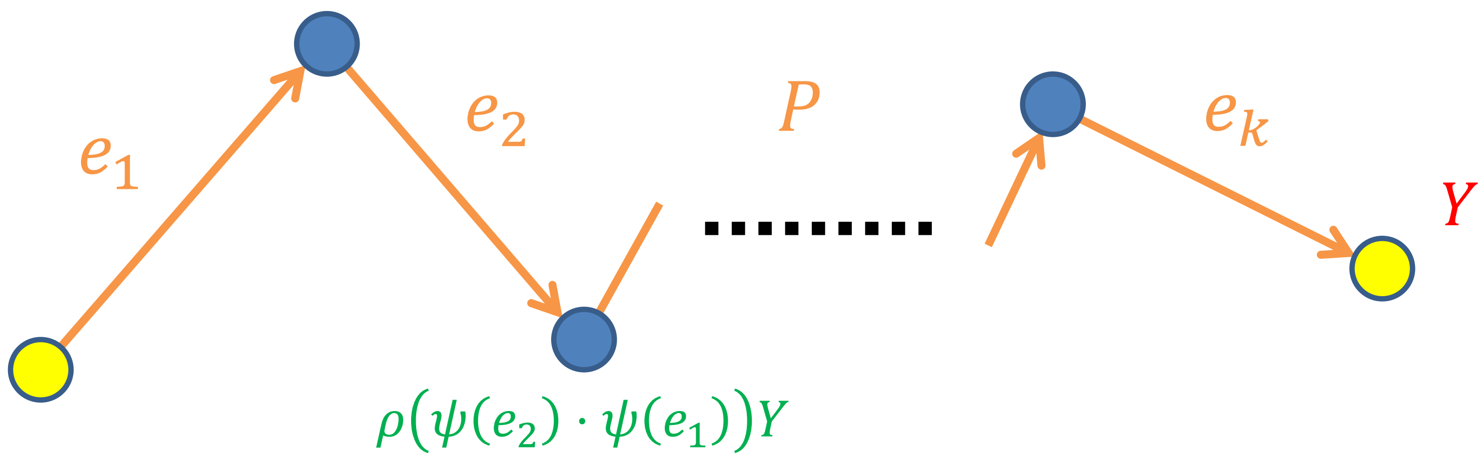

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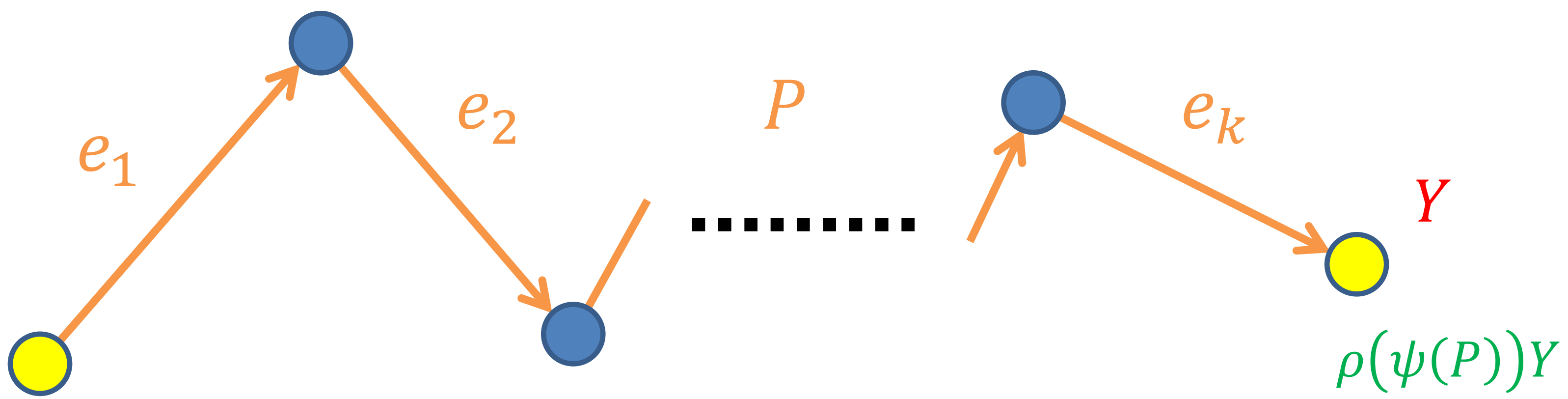

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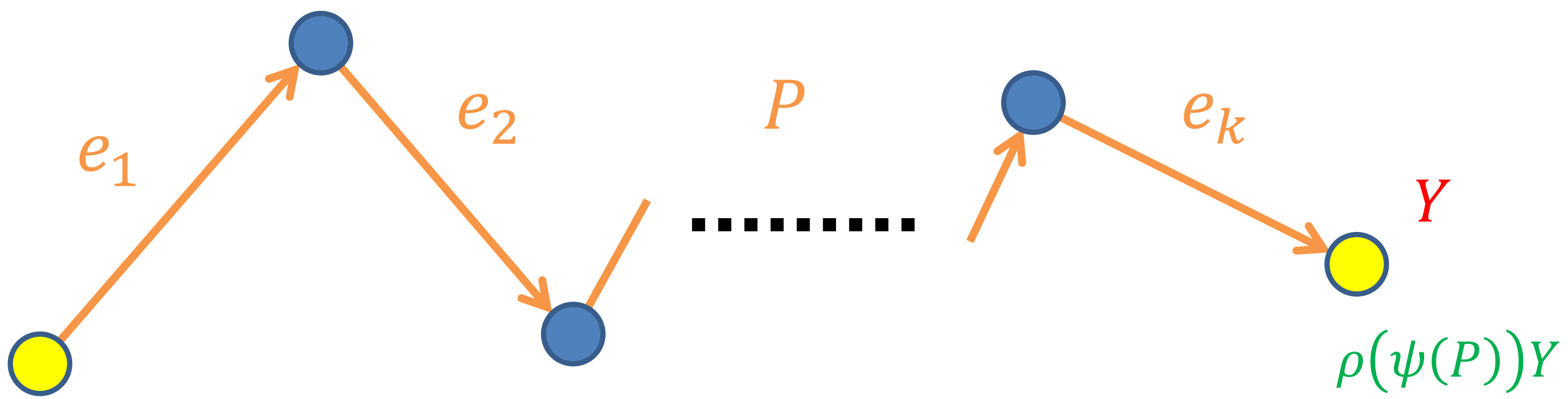


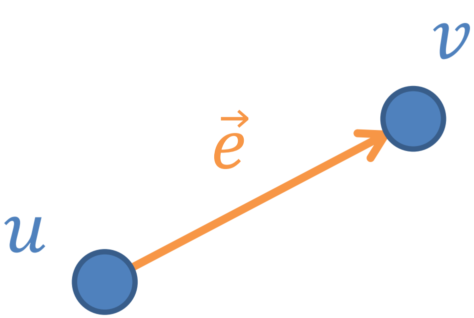
$\rho(\psi(P))Y = Y \Leftrightarrow \psi(P) \in \Gamma'$

Linearly dependent Non-admissible

Construction of matrix (step 1)

$\rho: \Gamma \to \mathrm{PGL}(2, \mathbf{F})$ homomorphic, Y: 1-dim. subspace of $\mathbf{F}^2$
s.t. $\Gamma' = \{ \alpha \in \Gamma \mid \rho(\alpha) Y = Y \}$ ($\mathbf{F}$: field)

- Construct the incidence matrix of the input graph, where fix one direction $\vec{e} = uv \in \vec{E}$ of each edge $e \in E$.
- Replace (u, e)-entry by I_2 and (v, e)-entry by $-\rho(\psi(\vec{e}))$.



$$\begin{array}{c} \\ \\ u \\ \\ v \\ \\ \end{array} \begin{array}{c} e \\ \left[\begin{array}{c:c:c} & \mathbf{0} & \\ \hdashline & 1 & \\ \hdashline & \mathbf{0} & \\ \hdashline & -1 & \\ \hdashline & \mathbf{0} & \end{array}\right] \end{array}$$

Construction of matrix (step 1)

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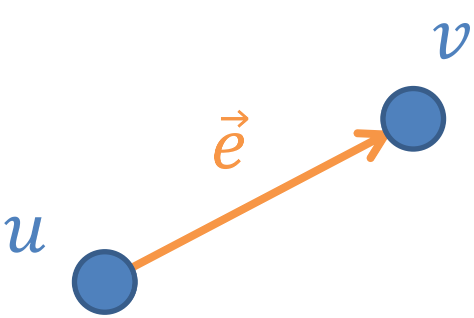


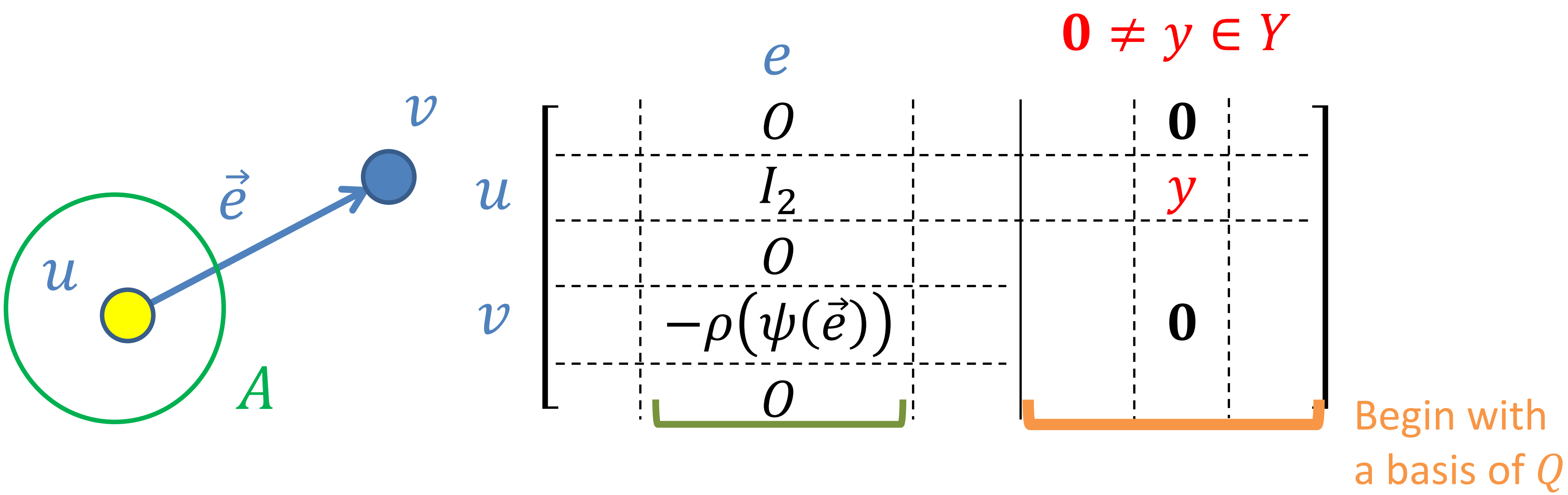
$$
\begin{array}{c} \\ u \\ \\ v \\ \\ \end{array}
\overset{e}{\begin{bmatrix} \mathbf{0} \\ 1 \\ \mathbf{0} \\ -1 \\ \mathbf{0} \end{bmatrix}}
\to
\begin{array}{c} \\ u \\ \\ v \\ \\ \end{array}
\overset{e}{\begin{bmatrix} O \\ I_2 \\ O \\ -\rho(\psi(\vec{e})) \\ O \end{bmatrix}}
$$

paired

Construction of matrix (step 2)

$\rho : \Gamma \to \mathrm{PGL}(2, \mathbf{F})$ homomorphic, Y: 1-dim. subspace of $\mathbf{F}^2$
s.t. $\Gamma' = \{\alpha \in \Gamma \mid \rho(\alpha)Y = Y\}$ ($\mathbf{F}$: field)

- $Q \coloneqq \{x \in (\mathbf{F}^2)^{V(G)} \mid x(v) \in Y \ (v \in A),\ x(v) = \mathbf{0} \ (v \notin A)\}$.
- The linear independence is considered in $(\mathbf{F}^2)^{V(G)}/Q$.



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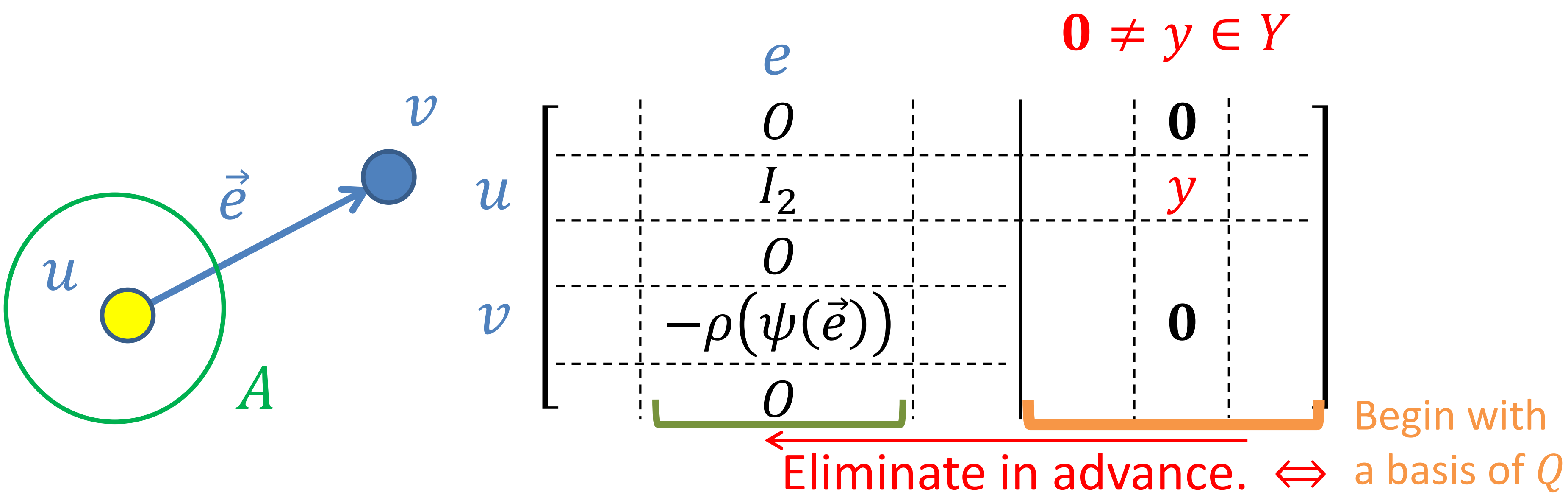
s.t. $\Gamma' = \{\alpha \in \Gamma \mid \rho(\alpha)Y = Y\}$ ($\mathbf{F}$: field)

- $Q \coloneqq \{x \in (\mathbf{F}^2)^{V(G)} \mid x(v) \in Y\ (v \in A),\ x(v) = \mathbf{0}\ (v \notin A)\}$.
- The linear independence is considered in $(\mathbf{F}^2)^{V(G)}/Q$.



Ex. 1. Packing odd-length A-paths

$\Gamma = (\{1, -1\}, \times)$, $\Gamma' = \{1\}$ $(\psi(e) \equiv -1)$

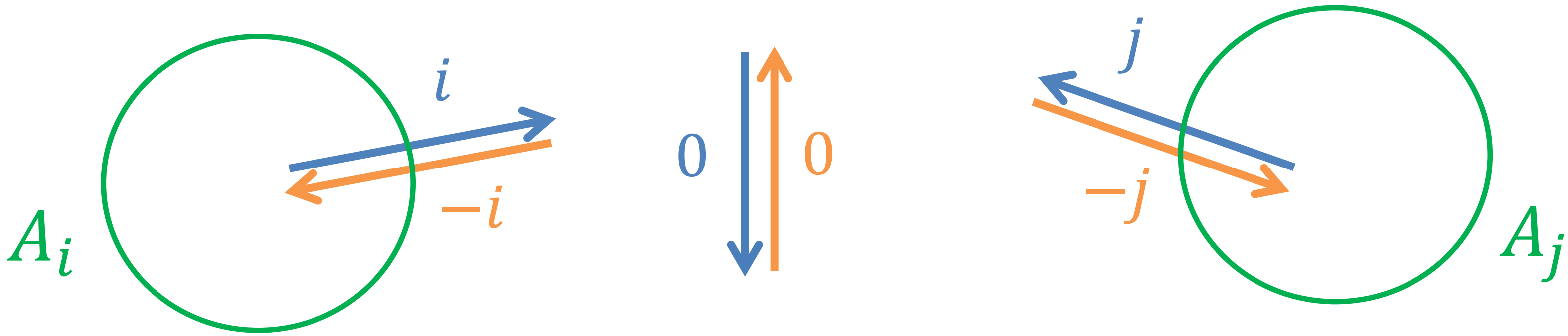
$\rightarrow$ $\rho(1) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $\rho(-1) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $Y = \left\langle \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\rangle$, $\mathbf{F}$: arbitrary

$$\rho(1)Y = \left\langle \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\rangle = \left\langle \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\rangle = Y$$

$$\rho(-1)Y = \left\langle \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\rangle = \left\langle \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\rangle \neq Y$$

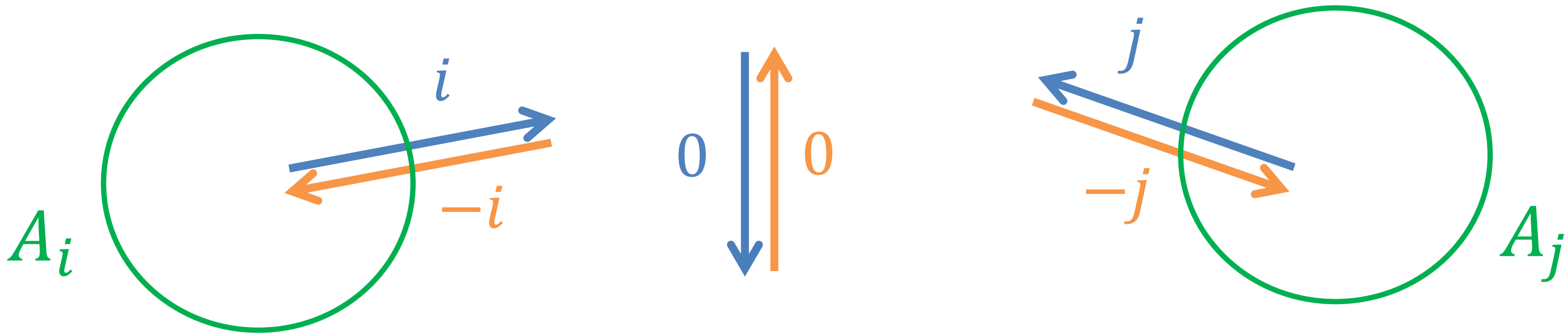
$\rho\colon \Gamma \rightarrow \mathrm{PGL}(2, \mathbf{F})$ homomorphic, Y: 1-dim. subspace of $\mathbf{F}^2$
s.t. $\Gamma' = \{\, \alpha \in \Gamma \mid \rho(\alpha)Y = Y \,\}$ ($\mathbf{F}$: field)

Ex. 2. Mader's $\mathcal{S}$-paths



$\Gamma = (\mathbf{Z}, +)$, $\Gamma' = \{0\}$ (ψ : as above)

Ex. 2. Mader's $\mathcal{S}$-paths



$\Gamma = (\mathbf{Z}, +)$, $\Gamma' = \{0\}$ (ψ : as above)

$\rightarrow$ $\rho(k) = \begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}$ $(k \in \mathbf{Z})$, $Y = \left\langle \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\rangle$, $\mathbf{F} = \mathbf{Q}$

$$\rho(k)Y = \left\langle \begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\rangle = \left\langle \begin{bmatrix} 1 \\ k \end{bmatrix} \right\rangle = Y \iff k = 0 \iff k \in \Gamma'$$

$\rho\colon \Gamma \rightarrow \mathrm{PGL}(2, \mathbf{F})$ homomorphic, Y: 1-dim. subspace of $\mathbf{F}^2$

s.t. $\Gamma' = \{ \alpha \in \Gamma \mid \rho(\alpha)Y = Y \}$ ($\mathbf{F}$: field)

Sufficient condition (again)

Main theorem (one direction)

$^{\exists}\rho\colon \Gamma \to \mathrm{PGL}(2, \mathbf{F})$ homomorphic, $^{\exists}Y$: 1-dim. subspace of $\mathbf{F}^2$
s.t. $\Gamma' = \{\, \alpha \in \Gamma \mid \rho(\alpha) Y = Y \,\}$

$\Rightarrow$ Subgroup model reduces to linear matroid parity.

Coherent representation

Main theorem

${}^{\exists}\rho\colon \Gamma \to \mathrm{PGL}(2, \mathbf{F})$ homomorphic, ${}^{\exists}Y$: 1-dim. subspace of $\mathbf{F}^2$
s.t. $\Gamma' = \{\, \alpha \in \Gamma \mid \rho(\alpha)Y = Y \,\}$

$\Leftrightarrow$ Subgroup model reduces to linear matroid parity
with coherent representation.

Coherent representation

Main theorem

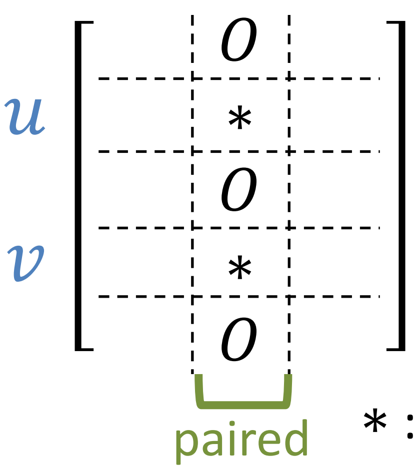
$^\exists\rho:\Gamma\to\mathrm{PGL}(2,\mathbf{F})$ homomorphic, $^\exists Y$: 1-dim. subspace of $\mathbf{F}^2$

s.t. $\Gamma' = \{\,\alpha\in\Gamma \mid \rho(\alpha)Y = Y\,\}$

$\Leftrightarrow$ Subgroup model reduces to linear matroid parity

with coherent representation.

$e = uv \in E$



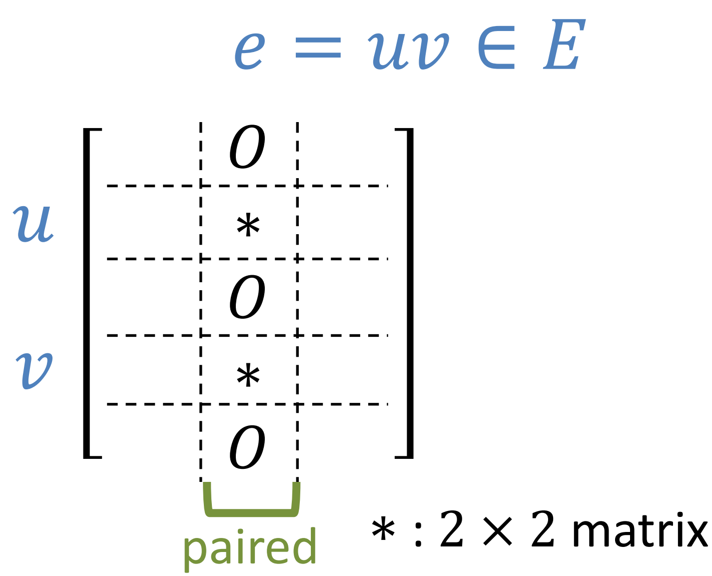
$*$: 2×2 matrix

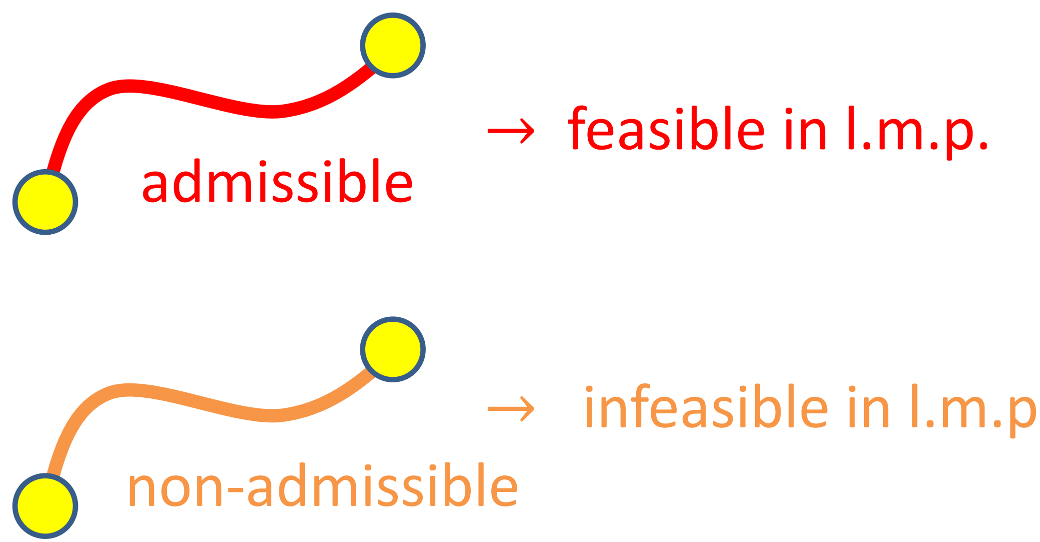
Coherent representation

Main theorem

${}^{\exists}\rho: \Gamma \to \mathrm{PGL}(2, \mathbf{F})$ homomorphic, ${}^{\exists}Y$: 1-dim. subspace of $\mathbf{F}^2$
s.t. $\Gamma' = \{\, \alpha \in \Gamma \mid \rho(\alpha) Y = Y \,\}$

$\Leftrightarrow$ Subgroup model reduces to linear matroid parity
with coherent representation.





Non-returning model

Non-returning model is formulated as subgroup model

$\Gamma = S_d$: symmetric group of degree $d \geq 2$,

$\Gamma' = S_{d-1} \quad (\sigma \in \Gamma' \Leftrightarrow \sigma(d) = d)$

Theorem (non-returning ver.)

Non-returning model admits our reduction $\Leftrightarrow d \leq 4$.

Remark. In the case of $d = 4$, ρ must be an isomorphism between S_4 and $\mathrm{PGL}(2, \mathbf{F}_3)$. $\quad (\mathbf{F}_3 = \mathbf{Z}/3\mathbf{Z})$

Conclusion

- Schrijver's reduction to linear matroid parity
is extendable to the subgroup model
of packing A-paths in group-labelled graphs, under some assumption on representability of the input groups.
The reduction leads to an $O(|E| + |V|^{\omega})$-time algorithm.

- For natural reduction with coherent representation,
the same assumption is necessary.

- Lovász's reduction idea to matroid matching
is always extendable.

(Tanigawa and Y., Packing non-zero A-paths via matroid matching, *METR* 2013-08, University of Tokyo. Available on the Web.)